CHAPTER 5 SERIES SOLUTIONS

1 Power Series Method

1.1 Power Series

$$\sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \ldots$$

where, a_0, a_1 , . . ., are constants (**coefficients**); x_0 is a constant (**center**)

Taylor's Formula

$$f(x) = \sum_{m=0}^{N} \frac{f^{(m)}(x_0)}{m!} (x - x_0)^m + R_N (x - x_0)$$

If $(x - x_0)$ is sufficiently small, $R_N (x - x_0) \to 0$ as $N \to \infty$, then, we say $f(x)$ is *analytic* at x_0, and

$$f(x) = \sum_{m=0}^{\infty} \frac{f^{(m)}(x_0)}{m!} (x - x_0)^m$$

Taylor Series

When $x_0 = 0$ $\Rightarrow$ ***Maclaurin Series***

Examples:

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

1.2 Basic Idea of the Power Series Method

In the previous discussion, the linear differential equations with constant coefficients were solved and shown to have solution for

$$y'' + ay' + by = 0$$

They can be anyone of the following 3 forms:

$$y = A_1 e^{m_1 x} + A_2 e^{m_2 x}$$
$$y = (A_1 + A_2 x)e^{mx}$$
$$y = e^{\alpha x}\left(A_1 \cos \beta x + A_2 \sin \beta x\right)$$

But, exponential, sine and cosine functions can be expressed in terms of Maclaurin series or Taylor series expanded around zero.

[Example] $y'' + y = 0$

[Solution] **Assume**

$$\boxed{y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \ldots}$$

$$y' = \sum_{m=1}^{\infty} m\, a_m x^{m-1} = a_1 + 2\, a_2 x + 3\, a_3 x^2 + \ldots; \qquad y'' = \sum_{m=2}^{\infty} m\,(m-1)\, a_m x^{m-2} = 2a_2 + 6\, a_3 x + 12\, a_4 x^2 + \ldots$$

Since $y'' + y = 0$

$\Rightarrow$ $(2a_2 + 6a_3 x + \ldots) + (a_0 + a_1 x + a_2 x^2 + \ldots) = 0$ or $(2a_2 + a_0) + (6a_3 + a_1)\, x + (12a_4 + a_2)\, x^2 + \ldots = 0$

Since **$1, x, x^2, \ldots, x^n$ are linearly independent functions**, we have

$2a_2 + a_0 = 0$	coefficients of x^0
$6a_3 + a_1 = 0$	coefficients of x^1
$12a_4 + a_2 = 0$	coefficients of x^2

$\therefore$ (1) $a_2, a_4, a_6, \ldots,$ can be expressed in terms of a_0 and (2) $a_3, a_5, a_7, \ldots,$ can be expressed in terms of a_1

where a_0 and a_1 are arbitrary constants. After solving the above simultaneous equations, we have

$$a_2 = -\frac{a_0}{2} = -\frac{a_0}{2!}; \quad a_3 = -\frac{a_1}{6} = -\frac{a_1}{3!}; \quad a_4 = \ldots = \frac{a_0}{4!}; \ldots$$

thus $$y = a_0 \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \right] + a_1 \left[x - \frac{x^3}{3!} + \ldots \right] = a_0 \cos x + a_1 \sin x$$

- Since every linear differential equation with constant coefficients always possesses a valid series solution, it is natural to expect the linear differential equations with variable coefficients to have series solutions too.

- Also, since the majority of series cannot be summed and written in a function form, it is to be expected that some solutions must be left in series form.

$$y'' + p(x)\, y' + q(x)\, y = 0$$

where $p(x)$ and $q(x)$ are expressed in polynomials.

We assume

$$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \ldots$$

$$y' = \sum_{m=1}^{\infty} m\, a_m x^{m-1} = a_1 + 2\, a_2 x + \ldots$$

$$y'' = \sum_{m=2}^{\infty} m\,(m-1)\, a_m x^{m-2} = 2\, a_2 + 3 \times 2 a_3 x + \ldots$$

(1) Put y, y' and y'' into the differential equation

(2) Collect terms of x^0, x^1, x^2, . . .,

(3) Solve a set of simultaneous equations of a_0, a_1, a_2, ...

2 Theory of Power Series Method

2.1 Introduction

Power Series:

$$S(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \ldots \qquad (1)$$

Partial Sum:

$$S_n(x) = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \ldots + a_n (x - x_0)^n \qquad (2)$$

Remainder:

$$R_n(x) = a_{n+1} (x - x_0)^{n+1} + a_{n+2} (x - x_0)^{n+2} + \ldots \qquad (3)$$

Note that $R_n = S - S_n$ or $| S_n - S | = | R_n |$

Convergence:

Definition 1: If $\lim_{n\to\infty} S_n(x_1) = S(x_1)$, then the series (1) converges at $x = x_1$ and $x_1 \neq x_0$

Definition 2: If the series converges, then for every given positive number ε (no matter how small, but not zero), we can find a number N such that $| S_n - S | < \varepsilon$ for every $n > N$

2.2 Radius of Convergence

Example 1:

$$\sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \ldots \Rightarrow \quad |x| > 1 \text{ divergent}; \quad |x| < 1 \quad \text{convergent}$$

Example 2:

$$\sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \ldots \; (= e^x) \quad \Rightarrow \quad \text{convergent for all } x.$$

If a series converges for all x in

$$| x - x_0 | < R$$

and diverges for

$$| x - x_0 | > R \qquad\qquad (0 < R < \infty)$$

then $\quad R = $ ***radius of convergence***

$R = \infty$ if series converges for all x.

R can be calculated by the following formula:

$$R = \frac{1}{\lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right|} \qquad \text{(Ratio Test)}$$

Ratio Test

$$\rho = \lim_{m\to\infty}\left|\frac{a_{m+1}(x-x_0)^{m+1}}{a_m(x-x_0)^m}\right| = \lim_{m\to\infty}\left|\frac{a_{m+1}(x-x_0)}{a_m}\right| = |x-x_0|\lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right|$$

if $\rho > 1$ divergent

$\rho < 1$ convergent

$\rho = 1$ test fails (i.e., inconclusive)

Since $\rho < 1$: convergence, we need $|x-x_0|\lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right| = \lim_{m\to\infty}\left|\frac{a_{m+1}(x-x_0)}{a_m}\right| < 1$

$$|x-x_0| < \frac{1}{\lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right|} = R$$

(radius of convergence)

[Example] $e^x = 1 + x + \frac{x^2}{2!} + \ldots = \sum_{m=0}^{\infty} \frac{x^m}{m!}$

$$\rho = \lim_{m\to\infty} \left| \frac{a_{m+1} x}{a_m} \right| = \lim_{m\to\infty} \left| \frac{x/(m+1)!}{1/m!} \right| = \lim_{m\to\infty} \frac{x}{m+1} = 0 < 1$$

$\Rightarrow$ The series converges, i.e.,

$$R = \lim_{m\to\infty} \frac{1/m!}{1/(m+1)!} = \lim_{m\to\infty} (m+1) = \infty,$$ i.e., converges for all x.

[Example] $\sum_{m=0}^{\infty} x^m = 1 + x + x^2 + x^3 + \ldots$

$$\rho = \lim_{m\to\infty} \left| \frac{x\, a_{m+1}}{a_m} \right| = \lim_{m\to\infty} |x| = |x|$$

thus, converges for $|x| < 1$
diverges for $|x| > 1$

test fails for $|x| = 1$

Radius of convergence

$|x| < R = \lim_{m\to\infty} \frac{1}{1} = 1$, i.e., converges for all x in $|x| < 1$.

In fact, this series converges to $\frac{1}{1-x}$ for $-1 < x < 1$.

[Example] $\sum_{m=0}^{\infty} m!\, x^m = 1 + x + 2x^2 + 6x^3 + \ldots$

$$\rho = \lim_{m\to\infty}\left|\frac{x\, a_{m+1}}{a_m}\right| = \frac{x\,(m+1)!}{m!} = x\,(m+1) = \infty > 1$$

$R = \lim_{m\to\infty}\left|\frac{a_m}{a_{m+1}}\right| = \lim_{m\to\infty}\frac{1}{m+1} = 0$ Thus, this series diverges for all $x \neq 0$.

[Example] $\sum_{m=0}^{\infty} \frac{(-1)^m}{8^m}\, x^{3m}$

This is a series in powers of $t = x^3$ with coefficients $a_m = \frac{(-1)^m}{8^m}$, so that $\rho = \lim_{m\to\infty}\left|\frac{t\, a_{m+1}}{a_m}\right| = \frac{|\,t\,|}{8}$

thus, converges for

$$\frac{|\,t\,|}{8} < 1 \quad \text{or} \quad |\,t\,| < 8 \quad \text{or} \quad R = \lim_{m\to\infty}\left|\frac{a_m}{a_{m+1}}\right| = 8,\ \text{i.e.,}\ |\,x\,| < 2$$

2.3 Properties of Power Series

(1) A power series may be differentiated term by term (***Term-wise Differentiation***).

$$y(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m, \quad |x - x_0| < R \text{ and } R > 0 \quad \Rightarrow \quad y'(x) = \sum_{m=0}^{\infty} m\, a_m (x - x_0)^{m-1} = \sum_{m=1}^{\infty} m a_m (x - x_0)^{m-1}$$

(2) Two power series may be added term by term (***Term-wise Addition***).

$$f(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m \quad \text{and} \quad g(x) = \sum_{m=0}^{\infty} b_m (x - x_0)^m \quad \Rightarrow \quad f(x) + g(x) = \sum_{m=0}^{\infty} (a_m + b_m)(x - x_0)^m$$

(3) Two power series may be multiplied term by term (***Term-wise Multiplication***).

$$\Rightarrow \quad f(x)\, g(x) = \sum_{m=0}^{\infty} (a_0 b_m + a_1 b_{m-1} + \ldots + a_m b_0)(x - x_0)^m$$

(4) Vanishing of all Coefficients (***Linearly Independence***).

$$f(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m = 0 \text{ for } \textit{all } x \text{ in } |x - x_0| < R \quad \Rightarrow \quad a_m = 0 \text{ for } \underline{\textit{all}} \text{ m.}$$

Let's ask ourselves a question: Can **all** linear second-order variable coefficient differential equations be solved by power series method? Let us answer this question by the following illustration:

[Example] Solve the following Euler equations

$$x^2 y'' + a x y' + b y = 0$$

where (i) $a = -2, \quad b = 2$

(ii) $a = -1, \quad b = 1$

(iii) $a = 1, \quad b = 1$

[Solution] we assume

$$y = \sum_{m=0}^{\infty} c_m x^m ; \quad y' = \sum_{m=1}^{\infty} m c_m x^{m-1} = \sum_{m=0}^{\infty} m c_m x^{m-1};$$

$$y'' = \sum_{m=2}^{\infty} m(m-1) c_m x^{m-2} = \sum_{m=0}^{\infty} m(m-1) c_m x^{m-2}$$

$\therefore \quad x^2 y'' + a x y' + b y =$

$$\sum_{m=0}^{\infty} [m(m-1) + a m + b] c_m x^m = 0$$

Note that $m(m-1) + a m + b = 0$ *is the characteristic equation for Euler equation.*

Case (i) $a = -2, \quad b = 2$

$$\Rightarrow \quad \sum_{m=0}^{\infty} (m^2 - 3m + 2)\, c_m x^m = 0$$

$$\therefore \quad (m^2 - 3m + 2)\, c_m = 0 \quad \text{or} \quad (m-2)(m-1)\, c_m = 0$$

$$\Rightarrow \quad c_m = 0 \text{ for all } m \neq 1 \text{ or } 2 \quad (c_0 = c_3 = c_4 = \cdots = 0)$$

$\Rightarrow \quad y = c_1 x + c_2 x^2$ **Same if solved with characteristic equation!!!**

Case (ii) $a = -1, \quad b = 1$

$$\Rightarrow \quad \sum_{m=0}^{\infty} (m^2 - 2m + 1)\, c_m x^m = 0$$

$$\therefore \quad (m-1)^2 c_m = 0$$

$$\Rightarrow \quad c_m = 0 \quad \text{for all } m \neq 1$$

$\Rightarrow \quad y = c_1 x$ **$\therefore$In this case, power series method yields only one solution: $y = c_1 x$.**

We need another linearly independent solution to get the general solution of the differential equation.

$\Rightarrow$ Reduction of order: let $y_2 = x u \quad \Rightarrow \quad x^3 u'' + x^2 u' = 0$

$\Rightarrow \quad u = c \ln|x| \quad \Rightarrow \quad y = A x + B x \ln|x|$ **(Same as before!!!)**

Case (iii) $a = 1, \qquad b = 1$

$$\Rightarrow \quad (m^2 + 1)\, c_m = 0$$

$$\Rightarrow \quad c_m = 0 \qquad \text{for all } m$$

i.e., the power series method **fails** completely, but why??

By the way, the general solution of Case (iii) is

$$y = A\cos(\ln|x|) + B\sin(\ln|x|)$$

2.4 Regular Point and Singular Point

Analytic Function: If g is a function defined on an interval I, containing a point x_0, we say that g is *analytic* at x_0 if g can be expanded in a power series about x_0 which has *a positive radius of convergence*.

A function f is *real analytic* on an open set D in the real line if for any x_0 in D one can write

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$
$$= a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots$$

in which the coefficients a_0, a_1, ... are real numbers and the series is convergent to $f(x$ for x in a neighborhood of x_0.

Alternatively, an analytic function is an infinitely differentiable function such that the Taylor series at any point x_0 in its domain

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

converges to $f(x)$ for x in a neighborhood of x_0 (in the mean-square sense). The set of all real analytic functions on a given set D is often denoted by $C^{\omega}(D)$.

- ***Any polynomial in x is analytic for all x.***
- ***Any rational function (ratio of polynomials) is analytic for all values of x which are not zeros of the denominator polynomial.***

Question: Are e^x, $\sqrt{x}$, and $\frac{1}{x}$ analytic at $x = 0$?

Theorem *(Existence of Power Series Solutions)*

If the function p, q, r in

$$y'' + p(x)\, y' + q(x)\, y = r(x)$$

are analytic at $x = x_0$, then every solution y(x) of the above equation is analytic at $x = x_0$ and can be represented by a power series of x - x_0 with radius of convergence $R > 0$, i.e. $y = \sum_{m=0}^{\infty} a_m (x - x_0)^m$

Definition: *Regular Point and Singular Point*

We call $x = 0$ a ***regular point*** (or *ordinary point*) of the differential equation

$$y'' + p(x)\, y' + q(x)\, y = 0$$

when both p(x) and q(x) are analytic at $x = 0$.

If $x = 0$ is not a regular point, it is called a ***singular point*** of the differential equation.

[Example]

$$x\, y'' + 2\, y' + x\, y \;=\; 0$$

$$y'' + \frac{2}{x}\, y' + y \;=\; 0$$

$\Rightarrow$ $x \;=\; 0$ is a singular point! $\therefore$ may give some trouble in power series method.

Although it is inappropriate, we nonetheless assume $y \;=\; \sum_{m=0}^{\infty} c_m\, x^m$

The differential equation becomes

$$\sum_{m=2}^{\infty} m\,(\,m-1\,)\, c_m\, x^{m-1} + \sum_{m=1}^{\infty} 2\, m\, c_m\, x^{m-1} + \sum_{m=0}^{\infty} c_m\, x^{m+1} \;=\; 0$$

Let $m = k+1 \;\Rightarrow\; \sum_{m=2}^{\infty} m\,(\,m-1\,)\, c_m\, x^{m-1} \;=\; \sum_{k=1}^{\infty} (\,k+1\,)\, k\, c_{k+1}\, x^k$

Let $m = k+1 \;\Rightarrow\; \sum_{m=1}^{\infty} 2\, m\, c_m\, x^{m-1} \;=\; \sum_{k=0}^{\infty} 2\,(\,k+1\,)\, c_{k+1}\, x^k \;=\; 2\, c_1 + \sum_{k=1}^{\infty} 2\,(\,k+1\,)\, c_{k+1}\, x^k$

Let $m = k-1 \;\Rightarrow\; \sum_{m=0}^{\infty} c_m\, x^{m+1} \;=\; \sum_{k=1}^{\infty} c_{k-1}\, x^k$

Thus we have

$$2\,c_1 + \sum_{k=1}^{\infty} \{[\,(k+1)\,k + 2\,(k+1)\,]\,c_{k+1} + c_{k-1}\}\; x^k = 0 \quad \text{or} \quad 2\,c_1 + \sum_{k=1}^{\infty} \{(k+1)(k+2)\,c_{k+1} + c_{k-1}\}\, x^k = 0$$

$$\therefore \quad c_1 = 0$$

$$c_{k+1} = \frac{-\,c_{k-1}}{(k+2)(k+1)} \quad \text{for } k \geq 1$$

$$\therefore \quad c_3 = c_5 = c_7 = \ldots = 0$$

$$c_2 = -\frac{c_0}{3!} \qquad c_4 = \frac{c_0}{5!}$$

$$\therefore \quad y = c_0 \left\{ 1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \ldots \right\} = c_0 \frac{\sin x}{x}$$

Only one solution is obtained! The other linearly independent solution can be obtained by the **method of reduction of order**:

$$\Rightarrow \quad y_2 = u \frac{\sin x}{x} \quad \left(u' = \frac{1}{y_1^2} e^{-\int p(x)dx} \text{ where } y_1 = \frac{\sin x}{x} \text{ and } p = \frac{2}{x} \right) \quad \Rightarrow \quad y_2 = \frac{\cos x}{x} \qquad \text{(Exercise!)}$$

$$\therefore \quad y = A \frac{\sin x}{x} + B \frac{\cos x}{x}$$

Note that $\dfrac{\cos x}{x} = x^{-1} \left\{ 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \ldots \right\}$.

This suggests that **we *may* try $y = x^r\,(c_0 + c_1 + c_2\,x^2 + \ldots)$** in the first place to obtain the second linearly independent solution.

3 Frobenius Method

3.1 General Concepts

$$y'' + p(x)\, y' + q(x)\, y = 0$$

If p(x), q(x) are analytic at x = 0 $\Rightarrow$ x = 0 is a **regular point**, two linearly independent exist . $\Rightarrow$ $\mathbf{y = \sum_{m=0}^{\infty} a_m x^m}$

If p(x), q(x) are not analytic at x = 0 $\Rightarrow$ $\boxed{\textbf{singular point}}$

For x = 0 is a singular point, rewrite the differential equation in the following form:

$$y'' + p(x)\, y' + q(x)\, y = y'' + \frac{b(x)}{x}\, y' + \frac{c(x)}{x^2}\, y = 0$$

If b(x), c(x) analytic at x =0 $\Rightarrow$ $\boxed{\textbf{regular singular point}}$, at least one solution exist with the following form

$\Rightarrow$ $$\mathbf{y = x^r \sum_{m=0}^{\infty} a_m x^m}$$ where r is a parameter which need to be determined. It can be positive or negative.

If b(x), c(x) not analytic at x=0 $\Rightarrow$ $\boxed{\textbf{irregular singular point}}$, a non-trivial solution may or may not exist.

Theorem 1 (Frobenius Method)

Any differential equation of the form

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0$$

*where $b(x)$ and $c(x)$ are analytic at $x = 0$ (a regular singular point) , has **at least one** solution of the form*

$$y = x^r \sum_{m=0}^{\infty} a_m x^m = x^r (a_0 + a_1 x + a_2 x^2 + \ldots),$$

where $a_0 \neq 0$ and r may be any number (real or complex).

x=0 regular singular point!!!

3.2 Indicial Equation

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0 \qquad \text{or} \qquad x^2 y'' + x\, b(x)\, y' + c(x)\, y = 0$$

Since b(x) and c(x) are **analytic**, i.e.,

$$b(x) = b_0 + b_1 x + b_2 x^2 + \ldots$$

$$c(x) = c_0 + c_1 x + c_2 x^2 + \ldots$$

We let

$$y = x^r \sum_{m=0}^{\infty} a_m x^m = x^r \left(a_0 + a_1 x + a_2 x^2 + \cdots \right)$$

$$y' = \sum_{m=0}^{\infty} (m + r)\, a_m x^{m+r-1} = x^{r-1} \sum_{m=0}^{\infty} (m + r)\, a_m x^m$$

$$= x^{r-1} [\, r\, a_0 + (r + 1)\, a_1 x + \ldots \,]$$

$$y'' = \sum_{m=0}^{\infty} (m + r)(m + r - 1)\, a_m x^{m+r-2} = x^{r-2} \sum_{m=0}^{\infty} (m + r)(m + r - 1)\, a_m x^m$$

$$= x^{r-2} [\, r(r - 1)\, a_0 + (r + 1)\, r\, a_1 x + \ldots \,]$$

Put y, y', y'', b(x), c(x) into the differential equation and collect terms of x^p, we have (for x^r terms)

$$[\, r\,(r-1) + b_0 r + c_0\,]\, a_0 \;=\; 0$$

Since $a_0 \neq 0$, we have

$$r\,(r-1) + b_0 r + c_0 \;=\; 0$$

Indicial Equation !!!

Two roots for r:

one root for $\qquad y_1 = x^r \sum_{m=0}^{\infty} a_m x^m$

another root $\qquad \Rightarrow$ Theorem 2 for y_2

Theorem 2 (Form of the Second Solution)

Case 1: r_1 and r_2 differ but not by an integer

$$y_1 = x^{r_1}\,(a_0 + a_1 x + a_2 x^2 + \ldots)$$

$$y_2 = x^{r_2}\,(A_0 + A_1 x + A_2 x^2 + \ldots)$$

Case 2: $r_1 = r_2 = r, \quad r = \frac{1}{2}(1 - b_0)$

$$y_1 = x^{r}\,(a_0 + a_1 x + a_2 x^2 + \ldots)$$

$$y_2 = \mathbf{y_1 \ln x} + x^{r}\,(A_1 x + A_2 x^2 + \ldots)$$

Case 3: r_1 and r_2 differ by a nonzero integer, where $r_1 > r_2$

$$y_1 = x^{r_1}\,(a_0 + a_1 x + a_2 x^2 + \ldots)$$

$$y_2 = \mathbf{k\, y_1 \ln x} + x^{r_2}\,(A_0 + A_1 x + A_2 x^2 + \ldots)$$

where $r_1 - r_2 > 0$ and **k may or may not be zero!!!**

Note that in Case 2 and Case 3, **the second linearly independent solution y_2 can also be obtained by reduction of order method** (i.e., by assuming $y_2 = u\, y_1$).

Case 1: r_1 and r_2 differ but not by an integer

[Example] $y'' + \dfrac{1}{4x} y' + \dfrac{1}{8x^2} y = 0, \quad x > 0$ (Euler Equation)

[Solution] $y = x^r (a_0 + a_1 x + a_2 x^2 + \ldots) = x^r \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^{m+r}$

$$y' = \sum_{m=0}^{\infty} (m+r)\, a_m x^{m+r-1}$$

$$y'' = \sum_{m=0}^{\infty} (m+r)(m+r-1)\, a_m x^{m+r-2}$$

$$\Rightarrow \quad \sum_{m=0}^{\infty} a_m \left[(r+m)(r+m-1) + \frac{1}{4}(r+m) + \frac{1}{8} \right] x^{r+m-2} = 0$$

For m = 0, $a_m \neq 0$ ($a_0 \neq 0$ by Theorem 1), we have the indicial equation:

$$r(r-1) + \frac{1}{4} r + \frac{1}{8} = 0 \quad \Rightarrow \quad r_1 = 1/4 \quad \text{and} \quad r_2 = 1/2$$

Note that in this case, $r_1 \neq r_2$ and $r_1 - r_2$ is not an integer.

$$r_2 - r_1 = \frac{1}{4}$$

For $r = \frac{1}{4}$, we have $y_1 = x^{1/4}(a_0 + a_1 x + a_2 x^2 + \ldots)$

$$\therefore \quad \sum_{m=0}^{\infty} a_m \left[\left(\frac{1}{4} + m\right)\left(\frac{1}{4} + m - 1\right) + \frac{1}{4}\left(\frac{1}{4} + m\right) + \frac{1}{8} \right] x^{\frac{1}{4} + m - 2} = 0 \quad \text{or} \quad \sum_{m=0}^{\infty} a_m \left(m^2 - \frac{m}{4}\right) x^{m - \frac{7}{4}} = 0$$

which is valid for all $x > 0$.

Thus, we have $a_m\, m \left[m - \frac{1}{4} \right] = 0$ for all m (=0, 1, 2, …)

$\Rightarrow$ for $m = 0$ $a_0 =$ arbitrary nonzero constant

but for $m = 1, 2, \ldots$ $a_m = 0$

$$\Rightarrow \quad y_1 = a_0 x^{1/4}$$

Similarly, for $r = 1/2$, we have

(by setting $y_2 = x^{1/2}(A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \ldots$, Exercise!)

$$\Rightarrow \quad y_2 = A_0 x^{1/2}$$

Hence, the general solution is

$$y = a_0 x^{1/4} + A_0 x^{1/2}$$

[Example] $y'' + \frac{1}{x} y' + \frac{1}{x^2} y = 0, \; x > 0$

[Solution] Letting $y = x^r \sum_{m=0}^{\infty} a_m x^m$, we have $\sum_{m=0}^{\infty} a_m \left[(r+m)(r+m-1) + (r+m) + 1 \right] x^{r+m-2} = 0$

The indicial equation is $(m = 0, a_0 \neq 0)$ $\quad r(r-1) + r + 1 = r^2 + 1 = 0$

$\therefore \quad r_1 = i, \quad r_2 = -i, \quad r_1 - r_2 = 2i$ is not an integer.

For $r = i$

$$\sum_{m=0}^{\infty} a_m \left[(i+m)(i+m-1) + (i+m) + 1 \right] x^{i+m-2} = 0$$

or $\quad a_m \left[(i+m)^2 + 1 \right] = 0 \quad$ or $\quad a_m \, m (m + 2i) = 0$

$\Rightarrow \quad m = 0 \qquad a_0 \neq 0$, i.e., **$a_0$ is an arbitrary nonzero constant**

$\quad\;\; m \neq 0 \qquad a_m = 0$

$\Rightarrow \quad y_1 = a_0 x^i = a_0 e^{i \ln x} = a_0 \left[\cos(\ln x) + i \sin(\ln x) \right] = \cos(\ln x) + i \sin(\ln x)$ $\qquad$ By taking $a_0 = 1$

For **$r = -i$,** we have (Exercise!) $\quad y_2 = x^{-i} = \cos(\ln x) - i \sin(\ln x)$

Since the linear combinations of solutions are also solutions of the linear differential equation, thus,

$$y_1^* = \frac{1}{2}(y_1 + y_2) = \cos(\ln x) \quad \text{and} \quad y_2^* = \frac{1}{2i}(y_1 - y_2) = \sin(\ln x) \quad \Rightarrow \quad y = c_1 \cos(\ln x) + c_2 \sin(\ln x)$$

Case 2, $r_1 = r_2 = r$, Double Roots

[Example] $y'' + y' + \dfrac{1}{4x^2}\, y = 0, \qquad x > 0$

[Solution] Letting $y = x^r \sum_{m=0}^{\infty} a_m x^m$, we have

$$\sum_{m=0}^{\infty} a_m (r+m)(r+m-1) x^{r+m-2} + \sum_{m=0}^{\infty} a_m (r+m) x^{r+m-1} + \sum_{m=0}^{\infty} \frac{1}{4} a_m x^{r+m-2} = 0$$

$$r+m-1 = r+k-2 \quad \Rightarrow \quad m = k-1$$

Since $\sum_{m=0}^{\infty} a_m (r+m) x^{r+m-1} = \sum_{k=1}^{\infty} a_{k-1} (r+k-1) x^{r+k-2} = \sum_{m=1}^{\infty} a_{m-1} (r+m-1) x^{r+m-2}$

The differential equation becomes

$$a_0 \left[r(r-1) + \frac{1}{4} \right] x^{r-2} + \sum_{m=1}^{\infty} \left\{ a_m \left[(r+m)(r+m-1) + \frac{1}{4} \right] + a_{m-1} (r+m-1) \right\} x^{r+m-2} = 0$$

The indicial equation is

$$r(r-1) + \frac{1}{4} = 0 \quad \text{or} \quad \left[r - \frac{1}{2} \right]^2 = 0 \quad \Rightarrow \quad r_1 = r_2 = r = \frac{1}{2}$$

For $\mathbf{r} = \frac{1}{2}$

$$\sum_{m=1}^{\infty} \left\{ a_m\left[\left(\frac{1}{2} + m\right)\left(m - \frac{1}{2}\right) + \frac{1}{4} \right] + a_{m-1}\left(m - \frac{1}{2}\right) \right\} x^{m-\frac{3}{2}} = 0 \quad \text{or} \quad \sum_{m=1}^{\infty} \left\{ m^2 a_m + \left[m - \frac{1}{2} \right] a_{m-1} \right\} x^{m-\frac{3}{2}} = 0$$

$$\Rightarrow \qquad \mathbf{a_m = -\frac{\left[m - \frac{1}{2}\right] a_{m-1}}{m^2}} \quad \text{for } m \ge 1 \quad \textbf{(recurrence formula)}$$

$$\text{Hence} \qquad a_1 = \frac{a_0}{2}, \quad a_2 = -\frac{(3/2)a_1}{2^2} = -\frac{3}{2\cdot 2^2}\left(-\frac{a_0}{2}\right) = \frac{3\,a_0}{2^2\,2^2}, \ldots$$

$$\text{and } y_1 = a_0\, x^{\frac{1}{2}} \left\{ 1 - \frac{x}{2} + \frac{3}{2^2}\left[\frac{x}{2}\right]^2 + \ldots \right\} = x^{1/2} \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^3}\left(-\frac{x}{4}\right)^n, \qquad x > 0$$

Note that we have set $a_0 = 1$ in the above equation.

Approach 1

Since $r = r_1 = r_2$, another solution can be obtained by directly letting

$$y_2 = y_1 \ln x + x^r (A_1 x + A_2 x^2 + \ldots) \qquad \text{(Exercise!)}$$

$$\Rightarrow \qquad y_2 = y_1 \ln x + x^{\frac{1}{2}} \left[-\frac{x^2}{16} + \ldots \right]$$

Approach 2

We can also use the *method of reduction of order* to produce the second linearly independent solution, y_2, by letting

$$y_2 = u\, y_1$$

Put into the differential equation,

$$u''\, y_1 + u'\,(2\, y_1' + y_1) = 0 \quad \Rightarrow \quad \frac{u''}{u'} = -2\frac{y_1'}{y_1} - 1 = \ldots \text{(long division)} = -\frac{1}{x} - \frac{x}{4} + \ldots$$

i.e.,
$$\ln u' = -\ln x - \frac{x^2}{8} + \ldots \quad \text{or} \quad u' = \frac{1}{x}\exp\left\{-\frac{x^2}{8} + \ldots\right\} = \ldots$$

By expanding the exponential function in Taylor series and then integrating

$$u = \ln x - \frac{x^2}{16} + \ldots$$

$$\Rightarrow \quad y_2 = y_1\, u = y_1\left[\ln x - \frac{x^2}{16} + \ldots\right] = y_1 \ln x + \sqrt{x}\left[-\frac{x^2}{16} + \ldots\right]$$

Both are tedious and intractable!

[Exercise] $\quad x\, y'' + (1 - x)\, y' - y = 0, \; x > 0$

Case 3: r_1 and r_2 differ by an nonzero integer, $r_1 > r_2$

[Example] $x^2 y'' + x y' + \left[x^2 - \frac{1}{4} \right] y = 0$ (Bessel's equation of order 1/2)

[Solution] Put $y = x^r \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^{m+r}$

The differential equation becomes

$$\sum_{m=0}^{\infty} a_m (r+m)(r+m-1) x^{r+m-2} + \sum_{m=0}^{\infty} a_m (r+m) x^{r+m-2} - \sum_{m=0}^{\infty} \frac{1}{4} a_m x^{r+m-2} + \sum_{m=0}^{\infty} a_m x^{r+m} = 0$$

After substituting $\sum_{m=2}^{\infty} a_{m-2} x^{r+m-2}$ for the last term of the lhs of the above equation, we have

$$a_0 \left[r(r-1) + r - \frac{1}{4} \right] x^{r-2} + a_1 \left[r(r+1) + (r+1) - \frac{1}{4} \right] x^{r-1} + \sum_{m=2}^{\infty} \left\{ a_m \left[(r+m)(r+m-1) + (r+m) - \frac{1}{4} \right] + a_{m-2} \right\} x^{r+m-2} = 0$$

Thus, we have the indicial equation:

$$r(r-1) + r - \frac{1}{4} = r^2 - \frac{1}{4} = 0 \qquad \text{or} \qquad r_1 = \frac{1}{2} \qquad r_2 = -\frac{1}{2}$$

Note that $r_1 - r_2 = 1$ is *an integer*!!!

Note also that, when m=1, $r_2(r_2+1) + (r_2+1) - \frac{1}{4} = \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) + \frac{1}{2} - \frac{1}{4} = 0!$

For $r_1 = \frac{1}{2} \Rightarrow 2a_1 x^{-1/2} + \sum_{m=2}^{\infty} \left[a_m (m^2 + m) + a_{m-2} \right] x^{m - \frac{3}{2}} = 0$

$\therefore \quad a_1 = 0$ and $a_m = -\frac{a_{m-2}}{m(m+1)}$ for $m \geq 2 \quad \Rightarrow \quad y_1 = \frac{\sin x}{\sqrt{x}}$

For $r_2 = -\frac{1}{2}$, **both a0 and a1 are arbitrary!**

$$\sum_{m=2}^{\infty} \left[a_m (m^2 - m) + a_{m-2} \right] x^{m - \frac{5}{2}} = 0$$

$\therefore \quad a_m = -\frac{a_{m-2}}{m(m-1)}$

or $\quad a_2 = -a_0/2! \qquad a_4 = a_0/4! \qquad a_6 = -a_0/6!$

$\quad a_3 = -a_1/3! \qquad a_5 = a_1/5! \quad \ldots$

$\Rightarrow \quad y_2 = \frac{1}{\sqrt{x}} (a_0 \cos x + a_1 \sin x) \qquad \therefore$ The linearly independent solution is $\frac{\cos x}{\sqrt{x}}$

(Alternatively, the linearly independent solution y_2 can also be obtained by reduction of order method.)

$\Rightarrow \quad y = A \frac{\sin x}{\sqrt{x}} + B \frac{\cos x}{\sqrt{x}}$

$\Rightarrow$ **Note that k=0 in this case!**

[Example] $x^2 y'' + x y' + (x^2 - 1) y = 0$ (Bessel's equation of order 1)

[Solution] Letting $y = x^r \sum_{m=0}^{\infty} a_m x^m$, we have

$$a_0 [r (r - 1) + r - 1] x^{r-2} + a_1 [r (r + 1) + (r + 1) - 1] x^{r-1} + \sum_{m=2}^{\infty} \left(a_m ((r + m) (r + m - 1) + (r + m) - 1) - a_{m-2} \right) x^{r+m-2} = 0$$

$\therefore$ The indicial equation is $r (r - 1) + r - 1 = r^2 - 1 = 0$ $\therefore$ $r_1 = 1$ $r_2 = -1$

$$r_1 - r_2 = 2$$
$$r_2 (r_2 + 1) + (r_2 + 1) - 1 = -1 \neq 0$$
$$\therefore k \neq 0$$

For **r = 1**, we have $3 a_1 + \sum_{m=2}^{\infty} \left(a_m (m^2 + 2 m) + a_{m-2} \right) x^{m-1} = 0$

$\therefore$ $a_1 = 0$ and $a_m = -\dfrac{a_{m-2}}{m (m + 2)}$ for $m \geq 2$

$$\Rightarrow \quad y_1(x) = x \left\{ 1 - \frac{1}{1!\,2!} \left[\frac{x}{2} \right]^2 + \frac{1}{2!\,3!} \left[\frac{x}{2} \right]^4 - \ldots \right\}$$

For **r = – 1,** we have $$-a_1 x^{-2} + \sum_{m=2}^{\infty} \{ a_m (m^2 - 2m) + a_{m-2} \} x^{m-3} = 0$$

$$\Rightarrow \quad a_1 = 0 \quad \text{and} \quad m (m - 2) a_m = - a_{m-2} \quad \text{for } m \geq 2$$

But **for m = 2, we have 0 = a_0 which is not true.** Thus we ***can not*** obtain the second linearly independent solution by setting $y = x^r \sum_{m=0}^{\infty} a_m x^m$ with $r = -1$.

Approach 1

From the theorem, we need to *directly* assume that the second solution is of the form:

$$y_2 = \mathbf{k}\, y_1 \ln x + x^{r_2} (A_0 + A_1 x + A_2 x^2 + \ldots) = \frac{1}{4} y_1 \ln x - \frac{1}{2} x^{-1} + \frac{x}{16} + \ldots$$ **(Exercise!)**

Approach 2

Note that the second linearly independent solution can also be obtained by the method of reduction of order (Exercise!):

$$y_2 = u\, y_1 \Rightarrow \frac{u''}{u'} = \frac{-2 y_1'}{y_1} - \frac{1}{x} = \frac{-3}{x} + \frac{x}{2} + \ldots$$

$$\therefore \ln u' = -3 \ln x + \frac{x^2}{4} + \ldots$$

$$\Rightarrow u' = x^{-3} \exp\left\{ \frac{x^2}{4} + \ldots \right\} = x^{-3} + \frac{1}{4} x^{-1} + \ldots \quad \text{or} \quad u = -\frac{1}{2} x^{-2} + \frac{1}{4} \ln x + \ldots$$

[Exercise] $x y'' + (x - 1) y' - 2 y = 0$

4 Legendre's Equation

4.1 Legendre's Differential Equation

$$(1-x^2)\,y'' - 2xy' + n(n+1)\,y = 0$$

where n is any non-negative real number. Since n(n+1) is unchanged when n is replaced by –(n+1), then

(1) solution of n = n′ (where $n' \geq 0$) is the same as n = - (n′+1);

(2) solution of n = - n″ (where $n'' \geq 1$) is the same as n = n″-1.

The above equation can be written as $\quad y'' - \dfrac{2x}{1-x^2}\,y' + \dfrac{n(n+1)}{1-x^2}\,y = 0$

But $\quad \dfrac{1}{1-x^2} = 1 + x^2 + x^4 + \ldots \quad$ which is analytic at **x = 0** (**regular** point!).

Therefore, we can solve the above equation by assuming

$$y = \sum_{m=0}^{\infty} a_m x^m$$

$\Rightarrow$ **Recurrence formula**

$$a_{m+2} = -\frac{(n-m)(n+m+1)}{(m+2)(m+1)} a_m, \quad m = 0, 1, \ldots \quad \textbf{(Exercise!)}$$

or a_0 a_1

$$a_2 = -\frac{n(n+1)}{2!} a_0 \qquad a_3 = -\frac{(n-1)(n+2)}{3!} a_1$$

$$a_4 = \frac{(n-2)n(n+1)(n+3)}{4!} a_0$$

$$a_5 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!} a_1$$

. . .

$\therefore$ The general solution is $y = a_0 y_1 + a_1 y_2$

where

$$y_1(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \frac{n(n-2)(n-4)(n+1)(n+3)(n+5)}{6!}x^6 + \cdots$$

$$y_2(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 - \frac{(n-1)(n-3)(n-5)(n+2)(n+4)(n+6)}{7!}x^7 + \cdots$$

If n = 0, 1, 2, . . . (non-negative integer), then one of the above two solutions is a polynomial!

$$y_1(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \frac{n(n-2)(n-4)(n+1)(n+3)(n+5)}{6!}x^6 + \cdots$$

$$y_2(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 - \frac{(n-1)(n-3)(n-5)(n+2)(n+4)(n+6)}{7!}x^7 + \cdots$$

$$n = 0 \qquad\qquad n = 2, 4, 6, \cdots$$

$$y_1(1) = 1 \qquad y_1(1) = (-1)^{\frac{n}{2}} \frac{2 \cdot 4 \cdot 6 \cdots n}{1 \cdot 3 \cdot 5 \cdots (n-1)}$$

$$n = 1 \qquad\qquad n = 1, 3, 5, \cdots$$

$$y_2(1) = 1 \qquad y_2(1) = (-1)^{\frac{n-1}{2}} \frac{2 \cdot 4 \cdot 6 \cdots (n-1)}{1 \cdot 3 \cdot 5 \cdots n}$$

Thus, let

$$y = c_1 P_n(x) + c_2 Q_n(x)$$

where $P_n(x)$ = Legendre *polynomials* [It is desirable that $P_n(1) = 1$]

$Q_n(x)$ = Legendre functions of the second kind converges in $-1<x<1$, but $Q_n(\pm 1)$ = unbounded (This is due to the fact that the Legendre equation is *not analytic* at $x=+1$ and $x=-1$!)

4.2 Legendre Polynomials $P_n(x)$

Since $$a_{m+2} = -\frac{(n-m)(n+m+1)}{(m+2)(m+1)} a_m \quad \text{for } m = 0, 1, \ldots$$

If n = non-negative **integer**,

$a_{m+2} = 0$ for $m = n$ $\Rightarrow a_{n+2} = 0$

i.e., $a_{n+2} = a_{n+4} = a_{n+6} = \ldots = 0$

when n = even, $y_1 \Rightarrow$ polynomial of degree n
n = odd, $y_2 \Rightarrow$ polynomial of degree n

These polynomials, each divided by an appropriate constant, are called the **Legendre polynomials $P_n(x)$**, which have the value $P_n(1) = 1$. In other words, let

$$P_n(x) = \begin{cases} \dfrac{y_1(x)}{y_1(1)} & \text{when } n \text{ is even} \\[2ex] \dfrac{y_2(x)}{y_2(1)} & \text{when } n \text{ is odd} \end{cases}$$

Or, we can **choose** the coefficient of x^n in the Legendre polynomials $P_n(x)$ as

$$a_n = 1 \qquad \text{if } n = 0$$

$$a_n = \frac{(2n)!}{2^n (n!)^2} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!} \; ; \qquad \text{if } n = 1, 2, \ldots$$

Then, we can obtain the other coefficients in $\mathbf{P_n(x)}$ with the **recurrence formula**

$$a_{n-2} = -\frac{n(n-1)}{2(2n-1)} a_n = \frac{-(2n-2)!}{2^n (n-1)!(n-2)!}$$

$$\ldots$$

$$a_{n-2m} = (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!}$$

Then, the Legendre polynomial of degree n, $P_n(x)$ is given by

$$P_n(x) = \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m} \qquad \text{where} \qquad M = \begin{cases} \dfrac{n}{2} & \text{when } n \text{ is even} \\[2ex] \dfrac{n-1}{2} & \text{when } n \text{ is odd} \end{cases}$$

[Example]

$$P_0(x)=1, \quad P_1(x)=x, \quad P_2(x)=\frac{1}{2}(3x^2-1), \quad P_3(x)=\frac{1}{2}(5x^3-3x), \quad P_4(x)=\frac{1}{8}\left(35x^4-30x^2+3\right), \quad P_5(x)=\frac{1}{8}\left(63x^5-70x^3+15x\right)$$

In all cases, $P_n(1) = 1$, and $P_n(-1) = (-1)^n$.

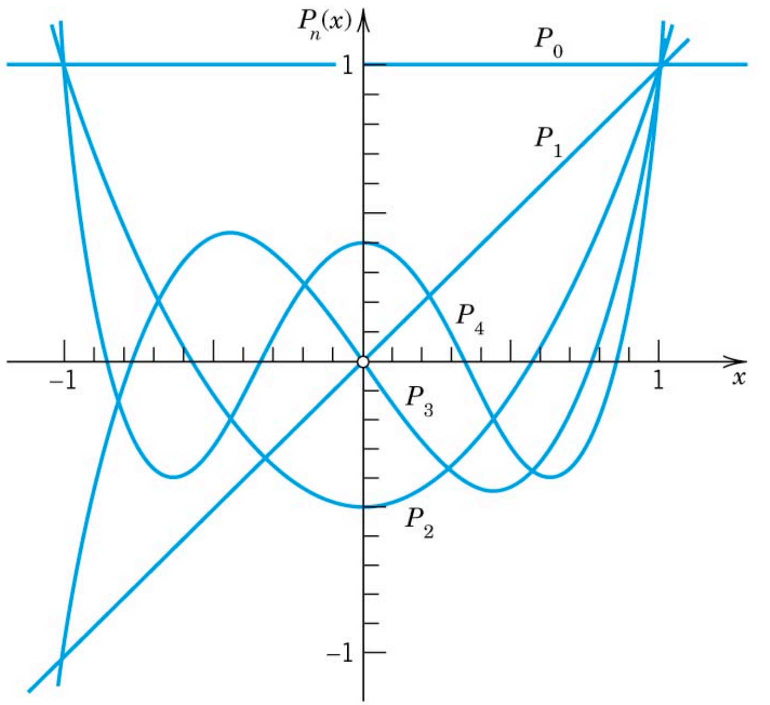


Fig. 101. Legendre polynomials

4.3 Legendre Functions of the Second Kind, $Q_n(x)$

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0, \quad n = 0, 1, 2, \ldots \quad \Rightarrow \quad y = c_1 P_n(x) + c_2 Q_n(x)$$

The power series $Q_n(x)$ can be obtained by the method of **reduction of order:**

$$\text{Let } Q_n(x) = u(x)P_n(x) = \begin{cases} -y_2(1)y_1(x) & \text{when } n \text{ is odd and } y_2(x) \text{ is a polynomial} \\ y_1(1)y_2(x) & \text{when } n \text{ is even and } y_1(x) \text{ is a polynomial} \end{cases}$$

$$Q_n'(x) = u'(x)P_n(x) + u(x)P_n'(x)$$
$$Q_n''(x) = u''(x)P_n(x) + 2u'(x)P_n'(x) + u(x)P_n''(x)$$

$$\begin{aligned}
&(1-x^2)Q_n''(x) - 2xQ_n'(x) + n(n+1)Q_n(x) \\
&= (1-x^2)\left[u''(x)P_n(x) + 2u'(x)P_n'(x) + u(x)P_n''(x)\right] \\
&\qquad -2x\left[u'(x)P_n(x) + u(x)P_n'(x)\right] + n(n+1)u(x)P_n(x) \\
&= (1-x^2)\left[u''(x)P_n(x) + 2u'(x)P_n'(x)\right] - 2xu'(x)P_n(x) \\
&= (1-x^2)u''(x)P_n(x) + u'(x)\left[2(1-x^2)P_n'(x) - 2xP_n(x)\right] = 0
\end{aligned}$$

$$\frac{u''(x)}{u'(x)} = -\left[2\frac{P_n'(x)}{P_n(x)} + \frac{-2x}{1-x^2}\right]$$

$$\ln u' = -\left[2\ln P_n(x) + \ln(1-x^2)\right] + c' \quad \Rightarrow u' = \frac{A_n}{[P_n(x)]^2(1-x^2)}$$

$$u(x) = A_n \int \frac{dx}{(1-x^2)[P_n(x)]^2} + B_n$$

$$Q_n(x) = u(x)P_n(x) = A_n P_n(x) \int \frac{dx}{(1-x^2)[P_n(x)]^2} + B_n P_n(x)$$

If $n=0$, then

$$P_0(x)=1$$

$$Q_0(x)=A_0\int\frac{dx}{1-x^2}+B_0$$

Also,

$$Q_0(0)=y_1(1)y_2(0)=0$$

$$Q_0'(0)=y_1(1)y_2'(0)=1$$

Thus

$$Q_0(x) = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| = x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots$$

If $n=1$, then

$$P_1(x)=x$$

$$Q_1(x)=A_1x\int\frac{dx}{(1-x^2)x^2}+B_1x$$

Also,

$$Q_1(0)=-y_2(1)y_1(0)=-1$$

$$Q_1'(0)=-y_2(1)y_1'(0)=0$$

Thus,

$$Q_1(x)=\frac{1}{2}x\ln\frac{1+x}{1-x}-1=xQ_0(x)-1 = x\left\{x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots\right\}-1$$

Note the most important property of $Q_n(x)$ is that $Q_n(\pm 1) =$ unbounded!!

4.4 Some Important Properties of $P_n(x)$

(1) Values of $P_n(x)$

$$P_n(1) = 1$$
$$P_n(-1) = (-1)^n$$
$$P_n(-x) = (-1)^n P_n(x)$$

n : even, $P_n(x)$: even function†

n : odd, $P_n(x)$: odd function

$$P_n'(-x) = (-1)^{n+1} P_n'(x)$$

(2) Rodrigues' Formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} [(x^2 - 1)^n]$$

[Exercise] Show that $P_2(x) = \frac{1}{2} (3 x^2 - 1)$

(3) Generating Function for Legendre Polynomials

$$\frac{1}{\sqrt{1 - 2 x t + t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n$$

(4) Recurrence Formulas

(i) $(n + 1) P_{n+1}(x) = (2 n + 1) x P_n(x) - n P_{n-1}(x), \quad n = 1, 2, \ldots$

(ii) $P_{n+1}'(x) - P_{n-1}'(x) = (2 n + 1) P_n(x)$

[Exercise] Starting with $P_0 = 1$, $P_1 = x$, derive $P_2, P_3, P_4, \ldots$ according to the recurrence formulas.

(5) Integrating Formulas

(i) $$\int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1} \qquad n = 0, 1, \ldots$$

(ii) $$\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = 0, \qquad m \neq n,\ m, n \in N$$

(6) Solution to

$$\frac{d^2y}{d\theta^2} + \cot\theta \frac{dy}{d\theta} + n(n+1)\,y = 0, \quad n = 0, 1, \ldots$$

is $y = c_1 P_n(\cos\theta) + c_2 Q_n(\cos\theta)$ ($x = \cos\theta$ and Exercise!)

Derivation of (4)(i)

Let $U(x,t) = 1 - 2xt + t^2$

$$U^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x) t^n$$

$$\frac{\partial}{\partial t} U^{-\frac{1}{2}} = -\frac{1}{2} U^{-\frac{3}{2}} (-2x + 2t) = \sum_{n=1}^{\infty} n P_n(x) t^{n-1}$$

$$(x - t) U^{-\frac{1}{2}} = U \sum_{n=1}^{\infty} n P_n(x) t^{n-1}$$

$$(x - t) \sum_{n=0}^{\infty} P_n(x) t^n = \left(1 - 2xt + t^2\right) \sum_{n=1}^{\infty} n P_n(x) t^{n-1}$$

$\Rightarrow$ Coefficients of t^n

$$x P_n(x) - P_{n-1}(x) = (n+1) P_{n+1}(x) - 2nx P_n(x) + (n-1) P_{n-1}(x)$$

$$(n+1) P_{n+1}(x) = (2n+1) x P_n(x) - n P_{n-1}(x)$$

Derivation of (4)(ii)

Let $U(x,t) = 1 - 2xt + t^2$

$$U^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x) t^n$$

$$\frac{\partial}{\partial x} U^{-\frac{1}{2}} = tU^{-\frac{3}{2}} = \sum_{n=1}^{\infty} P_n'(x) t^n$$

$$tU^{-\frac{1}{2}} = U \sum_{n=1}^{\infty} P_n'(x) t^n$$

$$t\sum_{n=0}^{\infty} P_n(x) t^n = \left(1 - 2xt + t^2\right) \sum_{n=1}^{\infty} P_n(x) t^n$$

$\Rightarrow$ Coefficients of t^{n+1}

$$P_n(x) = P_{n+1}'(x) - 2xP_n'(x) + P_{n-1}'(x) \qquad (*)$$

Differentiating (4)(i) wrt x, we have

$$(n+1) P_{n+1}'(x) = (2n+1) P_n(x) + (2n+1) xP_n'(x) - nP_{n-1}'(x)$$

Substituting (*) into the above equation yields

$$2(n+1) P_{n+1}'(x) = 2(2n+1) P_n(x) + (2n+1)\left[P_{n+1}'(x) + P_{n-1}'(x) - P_n(x)\right] - 2nP_{n-1}'(x)$$

$$\Rightarrow P_{n+1}'(x) - P_{n-1}'(x) = (2n+1) P_n(x)$$

Selecting the last coefficient in a Legendre polynomial

Let **a_n be the coefficient of x^n in $P_n(x)$**, i.e., its last coefficient, and **a_{n+1} be the coefficient of x^{n+1} in $P_{n+1}(x)$**, i.e., its last coefficient

$$(n+1)\,P_{n+1} - (2n+1)\,x\,P_n + n\,P_{n-1} = 0$$

The coefficient of x^{n+1} in LHS (a polynomial of degree n+1) of the above equation is given by

$$(n+1)\,a_{n+1} - (2n+1)\,a_n = 0 \qquad \therefore \qquad a_{n+1} = \frac{(2n+1)}{n+1}\,a_n$$

$$\therefore \qquad a_n = \frac{2n-1}{n}\,a_{n-1} = \frac{2n-1}{n}\,\frac{2n-3}{n-1}\,a_{n-2} = \ldots$$

$$= \frac{(2n-1)(2n-3)\cdots(5)(3)(1)}{n!}\,a_0$$

$$= \frac{\mathbf{(2n)!}}{\mathbf{2^n(n!)^2}}\,a_o$$

But a_0 is the coefficient of x^0 in $P_0(x) = 1$, we have $a_0 = 1$

$$\therefore \qquad a_n = \frac{(2n-1)(2n-3)\ldots(5)(3)(1)}{n!}$$

5 Bessel's Equations

5.0 Gamma Function, $\Gamma(\alpha)$ --- Appendix

Definition:

$$\Gamma(\alpha) \equiv \int_0^\infty e^{-t}\, t^{\alpha-1}\, dt$$

Properties:

(i) $\Gamma(\alpha+1) = \alpha\, \Gamma(\alpha)$

$$\Gamma(\alpha+1) = \int_0^\infty e^{-t}\, t^{\alpha}\, dt = \left[-e^{-t}\, t^{\alpha} \right]_0^\infty + \alpha \int_0^\infty e^{-t}\, t^{\alpha-1}\, dt = \alpha\, \Gamma(\alpha)$$

(i)' $\Gamma(\alpha) = \dfrac{\Gamma(\alpha+n)}{\alpha(\alpha+1)\ldots(\alpha+n-1)}$, $n \in N$

(ii) $\Gamma(1) = 1$ (from definition!) $\Gamma(2) = 1$, $\Gamma(3) = 2\,!, \ldots$

$\Gamma(n) = (n-1)\,!$, $n \in$ positive integer N

(iii) $\Gamma(1/2) = \sqrt{\pi}$ (Exercise! Hint: let $t^{1/2} = u$, $dt = 2u\, du$) ;

$\Gamma(3/2) = \dfrac{\sqrt{\pi}}{2}$

(iv) $\Gamma(-n) = \pm\infty$ $n \in N$

v) Plots of $\Gamma(x)$:

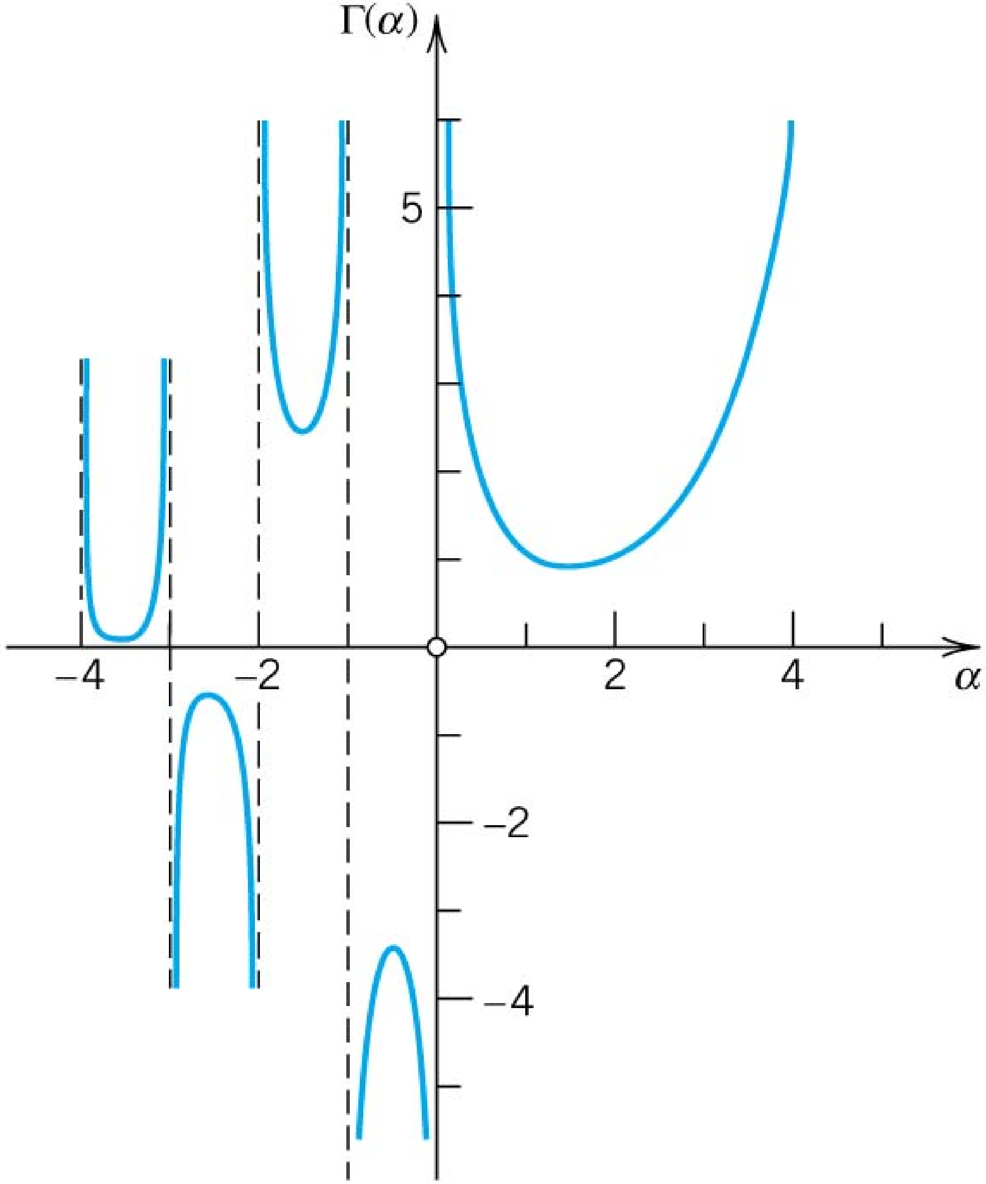


Fig. 517. Gamma function

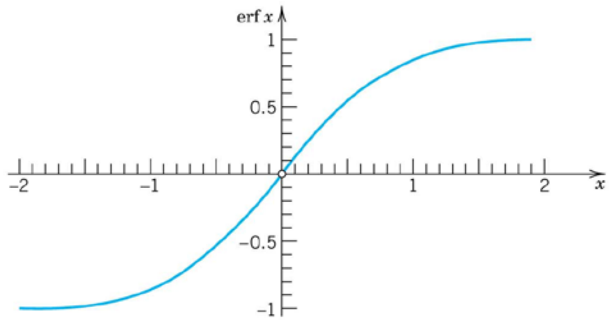


Fig. 518. Error function

Error function (Fig. 518 and Table A4 in Appendix 5)

(35)
$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

(36)
$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \left(x - \frac{x^3}{1!3} + \frac{x^5}{2!5} - \frac{x^7}{3!7} + - \cdots \right)$$

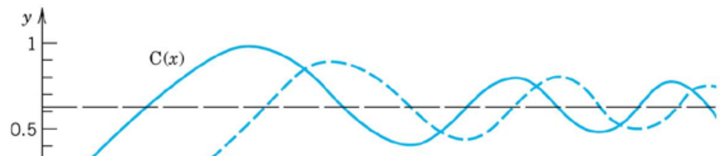
$\operatorname{erf}(\infty) = 1$, *complementary error function*

(37)
$$\operatorname{erfc} x = 1 - \operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt$$

Fresnel integrals[1] (Fig. 519)

(38)
$$C(x) = \int_0^x \cos(t^2)\, dt, \qquad S(x) = \int_0^x \sin(t^2)\, dt$$

$C(\infty) = \sqrt{\pi/8}$, $S(\infty) = \sqrt{\pi/8}$, *complementary functions*



5.1 Bessel's Differential Equation

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

where $\nu \geq 0$. Note that in this differential equation, $p(x) = x/x^2 = 1/x$ is not analytic at $x = 0$, thus we have to assume

$$y = \sum_{m=0}^{\infty} a_m x^{r+m}$$

⇒ Indicial Equation:

$$(r + \nu) (r - \nu) = 0 \quad \text{or} \quad r_1 = \nu \quad \text{and} \quad r_2 = - \nu$$

(i) Solution for $r_1 = \nu$

$$a_1 = 0 \qquad m = 1$$

$$a_m = \frac{-1}{(m + 2\nu)\, m}\, a_{m-2} \qquad m \geq 2$$

thus, $a_1 = a_3 = a_5 = \ldots = 0$

and $$a_2 = -\frac{a_0}{2^2\,(\nu + 1)}$$

$$a_4 = \frac{a_2}{4\,(2\nu + 4)} = \frac{a_0}{2^4\, 2!\,(\nu + 1)(\nu + 2)}$$

$$\ldots$$

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m}\, m!\,(\nu + 1)(\nu + 2)\ldots(\nu + m)} \quad ; \quad m = 1, 2, \ldots$$

$$\therefore \quad y_1(x) = a_0 x^\nu + a_0 x^\nu \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}\, m!\,(\nu + 1)(\nu + 2)\ldots(\nu + m)}$$

We let

$$a_0 = \frac{1}{2^\nu\, \Gamma(\nu + 1)}$$

$$\Rightarrow \quad y_1(x) = J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu}\, m!\, \Gamma(\nu + m + 1)}$$

= Bessel Function of the First Kind of Order ν

For $\nu = n \geq 0$, n : integer

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m! (n+m)!}$$

Note that

$$J_0(0) = 1$$
$$J_1(0) = J_2(0) = \ldots = 0$$
$$J_0(\infty) = J_1(\infty) = \ldots = 0$$

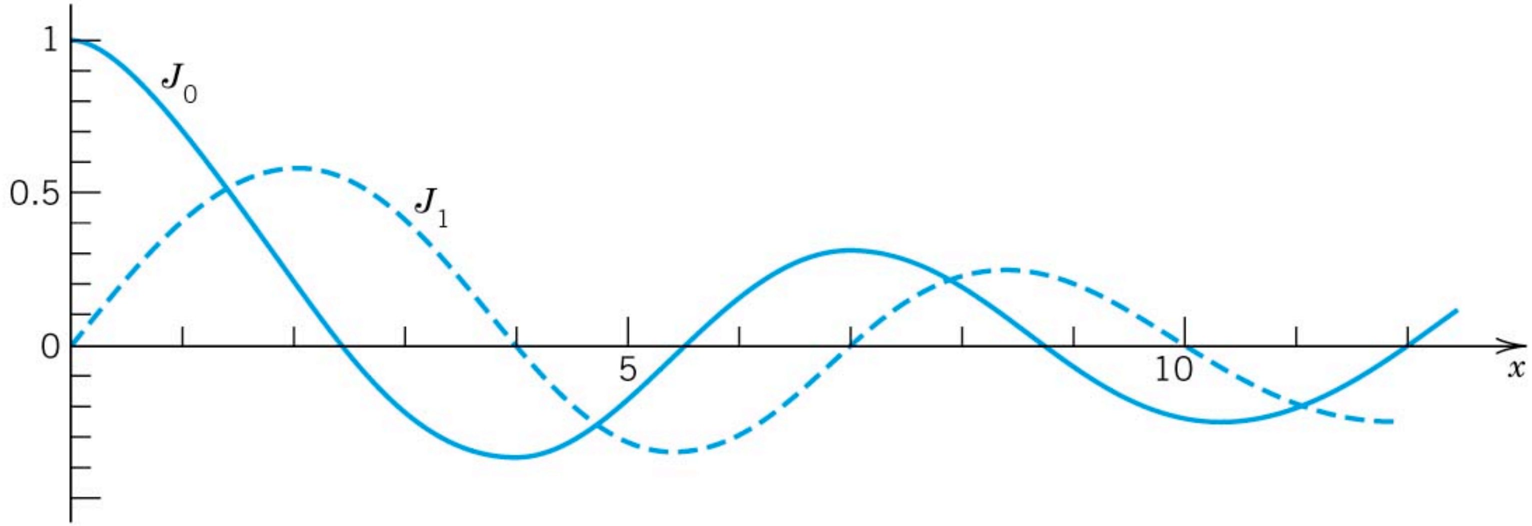


Fig. 103. Bessel functions of the first kind

APPENDIX 5

Tables

Tables of Laplace transforms in Secs. 5.8 and 5.9
Tables of Fourier transforms in Sec. 10.11

Table A1
Bessel Functions

For more extensive tables see Ref. [1] in Appendix 1.

x	$J_0(x)$	$J_1(x)$	x	$J_0(x)$	$J_1(x)$	x	$J_0(x)$	$J_1(x)$
0.0	1.0000	0.0000	3.0	−0.2601	0.3391	6.0	0.1506	−0.2767
0.1	0.9975	0.0499	3.1	−0.2921	0.3009	6.1	0.1773	−0.2559
0.2	0.9900	0.0995	3.2	−0.3202	0.2613	6.2	0.2017	−0.2329
0.3	0.9776	0.1483	3.3	−0.3443	0.2207	6.3	0.2238	−0.2081
0.4	0.9604	0.1960	3.4	−0.3643	0.1792	6.4	0.2433	−0.1816
0.5	0.9385	0.2423	3.5	−0.3801	0.1374	6.5	0.2601	−0.1538
0.6	0.9120	0.2867	3.6	−0.3918	0.0955	6.6	0.2740	−0.1250
0.7	0.8812	0.3290	3.7	−0.3992	0.0538	6.7	0.2851	−0.0953
0.8	0.8463	0.3688	3.8	−0.4026	0.0128	6.8	0.2931	−0.0652
0.9	0.8075	0.4059	3.9	−0.4018	−0.0272	6.9	0.2981	−0.0349
1.0	0.7652	0.4401	4.0	−0.3971	−0.0660	7.0	0.3001	−0.0047
1.1	0.7196	0.4709	4.1	−0.3887	−0.1033	7.1	0.2991	0.0252
1.2	0.6711	0.4983	4.2	−0.3766	−0.1386	7.2	0.2951	0.0543
1.3	0.6201	0.5220	4.3	−0.3610	−0.1719	7.3	0.2882	0.0826
1.4	0.5669	0.5419	4.4	−0.3423	−0.2028	7.4	0.2786	0.1096
1.5	0.5118	0.5579	4.5	−0.3205	−0.2311	7.5	0.2663	0.1352
1.6	0.4554	0.5699	4.6	−0.2961	−0.2566	7.6	0.2516	0.1592
1.7	0.3980	0.5778	4.7	−0.2693	−0.2791	7.7	0.2346	0.1813
1.8	0.3400	0.5815	4.8	−0.2404	−0.2985	7.8	0.2154	0.2014
1.9	0.2818	0.5812	4.9	−0.2097	−0.3147	7.9	0.1944	0.2192
2.0	0.2239	0.5767	5.0	−0.1776	−0.3276	8.0	0.1717	0.2346
2.1	0.1666	0.5683	5.1	−0.1443	−0.3371	8.1	0.1475	0.2476
2.2	0.1104	0.5560	5.2	−0.1103	−0.3432	8.2	0.1222	0.2580
2.3	0.0555	0.5399	5.3	−0.0758	−0.3460	8.3	0.0960	0.2657
2.4	0.0025	0.5202	5.4	−0.0412	−0.3453	8.4	0.0692	0.2708
2.5	−0.0484	0.4971	5.5	−0.0068	−0.3414	8.5	0.0419	0.2731
2.6	−0.0968	0.4708	5.6	0.0270	−0.3343	8.6	0.0146	0.2728
2.7	−0.1424	0.4416	5.7	0.0599	−0.3241	8.7	−0.0125	0.2697
2.8	−0.1850	0.4097	5.8	0.0917	−0.3110	8.8	−0.0392	0.2641
2.9	−0.2243	0.3754	5.9	0.1220	−0.2951	8.9	−0.0653	0.2559

$J_0(x) = 0$ for $x = 2.405, 5.520, 8.654, 11.792, 14.931, \cdots$
$J_1(x) = 0$ for $x = 0, 3.832, 7.016, 10.173, 13.324, \cdots$

(ii) Solution for $r_2 = -\nu$

$$r_1 - r_2 = 2\nu \qquad \begin{cases} \text{integer} \\ \text{non-integer} \end{cases}$$

Recall that

$$y_2 = k\, y_1(x) \ln x + x^{r_2} (A_0 + A_1 x + A_2 x^2 + \ldots)$$

$k = 1$, if $r_1 - r_2 = 0$

k <u>may</u> be zero if $r_1 - r_2 \neq 0$, but $r_1 - r_2 \in I$

(a) $x^2 y'' + x y' + [x^2 - (1/2)^2]\, y = 0$

$$r_1 = \frac{1}{2}, \; r_2 = -\frac{1}{2}, \; r_1 - r_2 = 1, \text{ but } \quad y_1 = \frac{\sin x}{\sqrt{x}}, \; y_2 = \frac{1}{\sqrt{x}} (a_0 \cos x + a_1 \sin x),$$

In this case, $k = 0$.

(b) $x^2 y'' + x y' + (x^2 - 1)\, y = 0$

$$r_1 = 1, \; r_2 = -1, \; r_1 - r_2 = 2, \text{ and } \quad y_1 = x \sum a_m x^m \; ; \; y_2 = \frac{1}{4} y_1 \ln x - \frac{1}{2} x^{-1} + \ldots$$

In this example, $k \neq 0$.

In general, substitute $r_1 = \nu$ into $y_1 = \sum_{m=0}^{\infty} a_m x^{m+r}$, we have

$$y_1 = J_\nu(x)$$

Similarly, substitution of $r_2 = -\nu$ into $y_2 = \sum_{m=0}^{\infty} a_m x^{m+r}$ gives

$$y_2 = J_{-\nu}(x)$$

Are $J_\nu(x)$ and $J_{-\nu}(x)$ linearly independent?

<u>Case 1</u> $\boldsymbol{\nu \notin N}$

$J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent. In this case, the general solution of the Bessel differential equation is

$$y = c_1 J_\nu(x) + c_2 J_{-\nu}(x), \qquad \nu \notin N$$

Case 2 $\nu \in \mathbf{N}$ $\{0, 1, 2, ...\}$

It can be shown that (Exercise! Also read p. 230, theorem 2 of the textbook.)

$$J_{-n}(x) = (-1)^n J_n(x)$$

i.e., J_{-n} and J_n are linearly dependent, we need to find the second linearly independent solution by assuming (or by method of reduction of order)

$$y_2 = k (\ln x) y_1 + x^{r_2} (A_0 + A_1 x + A_2 x^2 + ...), \quad x > 0$$

It yields (try this as an exercise!)

$$y_2 = J_n(x) \ln x - \frac{1}{2} \sum_{m=0}^{n-1} \frac{(n-m-1)!}{m!} \left(\frac{x}{2}\right)^{2m-n}$$

$$- \frac{1}{2} \frac{h_n}{n!} \left(\frac{x}{2}\right)^{n}$$

$$- \frac{1}{2} \sum_{m=1}^{\infty} \frac{(-1)^m [h_m + h_{m+n}]}{m!(n+m)!} \left(\frac{x}{2}\right)^{2m+n}$$

where $h_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$

It is customary to replace y_2 by the linear combination of solutions

$$Y_n(x) = \frac{2}{\pi} \{ y_2(x) + (\gamma - \ln 2) J_n(x) \}, \quad n = 0, 1, 2, \ldots$$

where $\gamma = \lim\limits_{n\to\infty} (h_n - \ln n) = 0.5772156649$ = Euler Constant.

Y_n is called the **Bessel function of the second kind of order n** or the **Neumann's function of order n**.

Thus, the general solution for $\nu = n \in N$ is

$$y(x) = c_1 J_n(x) + c_2 Y_n(x)$$

Plots of Y_0 and Y_1 vs. x can be found in Fig. 105, p. 231 of the textbook. Note that for all integer n, $Y_n(0) = -\infty$ and $Y_n(\infty) = 0$.

The function Y_n can be extended to all real numbers $\nu \geq 0$ by letting

$$Y_\nu(x) = \frac{1}{\sin\nu\pi} [J_\nu(x) \cos \nu\pi - J_{-\nu}(x)], \qquad \nu \neq 0, 1, 2\text{-}0$$

$$Y_n(x) = \lim_{\nu\to n} Y_\nu(x)$$

In general, the general solution of Bessel's equation of order ν (regardless whether ν is integer or not) can be written as

$$y(x) = c_1 J_\nu(x) + c_2 Y_\nu(x)$$

[Example] $x^2 y'' + x y' + (x^2 - 1/4) y = 0$

[Solution] $y = c_1 J_{1/2}(x) + c_2 Y_{1/2}(x)$

or $y = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x)$

(You have to specify what $J_{1/2}$, $Y_{1/2}$, $J_{-1/2}$ are in your answer.)

[Example] $x^2 y'' + x y' + (x^2 - 1) y = 0$

[Solution] $y = c_1 J_1(x) + c_2 Y_1(x)$

but **not** $y = c_1 J_1(x) + c_2 J_{-1}(x)$

since $J_1(x)$ and $J_{-1}(x)$ are linearly dependent!

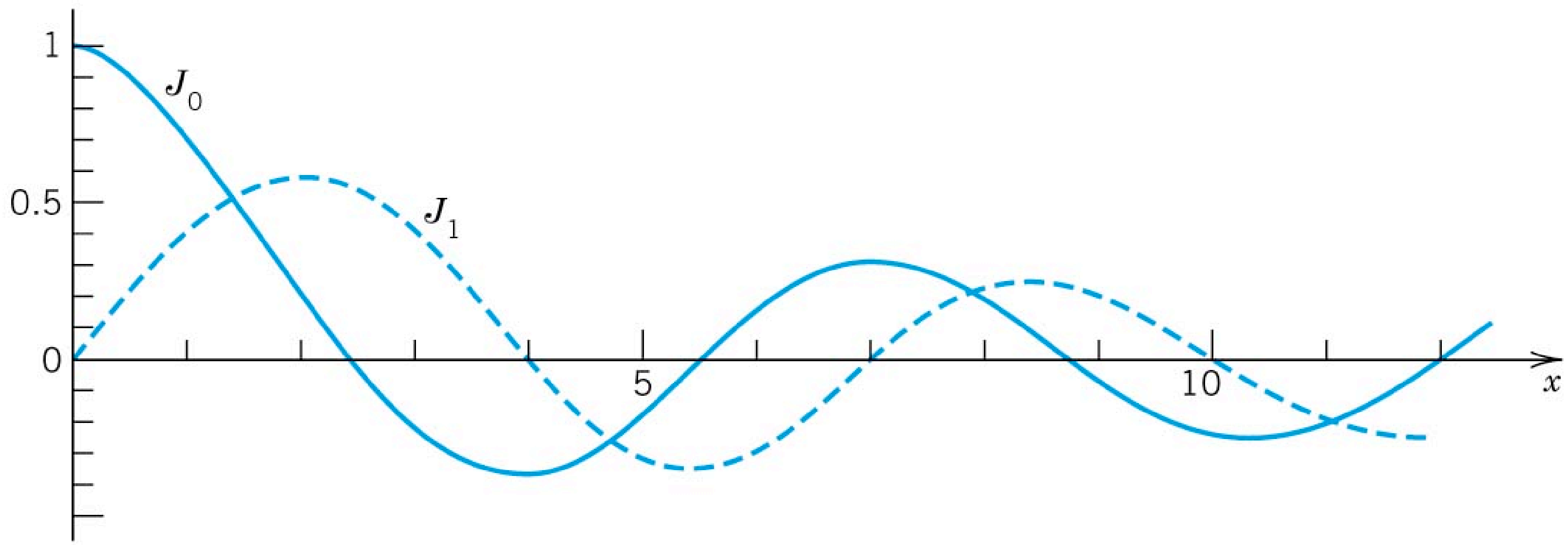


Fig. 103. Bessel functions of the first kind

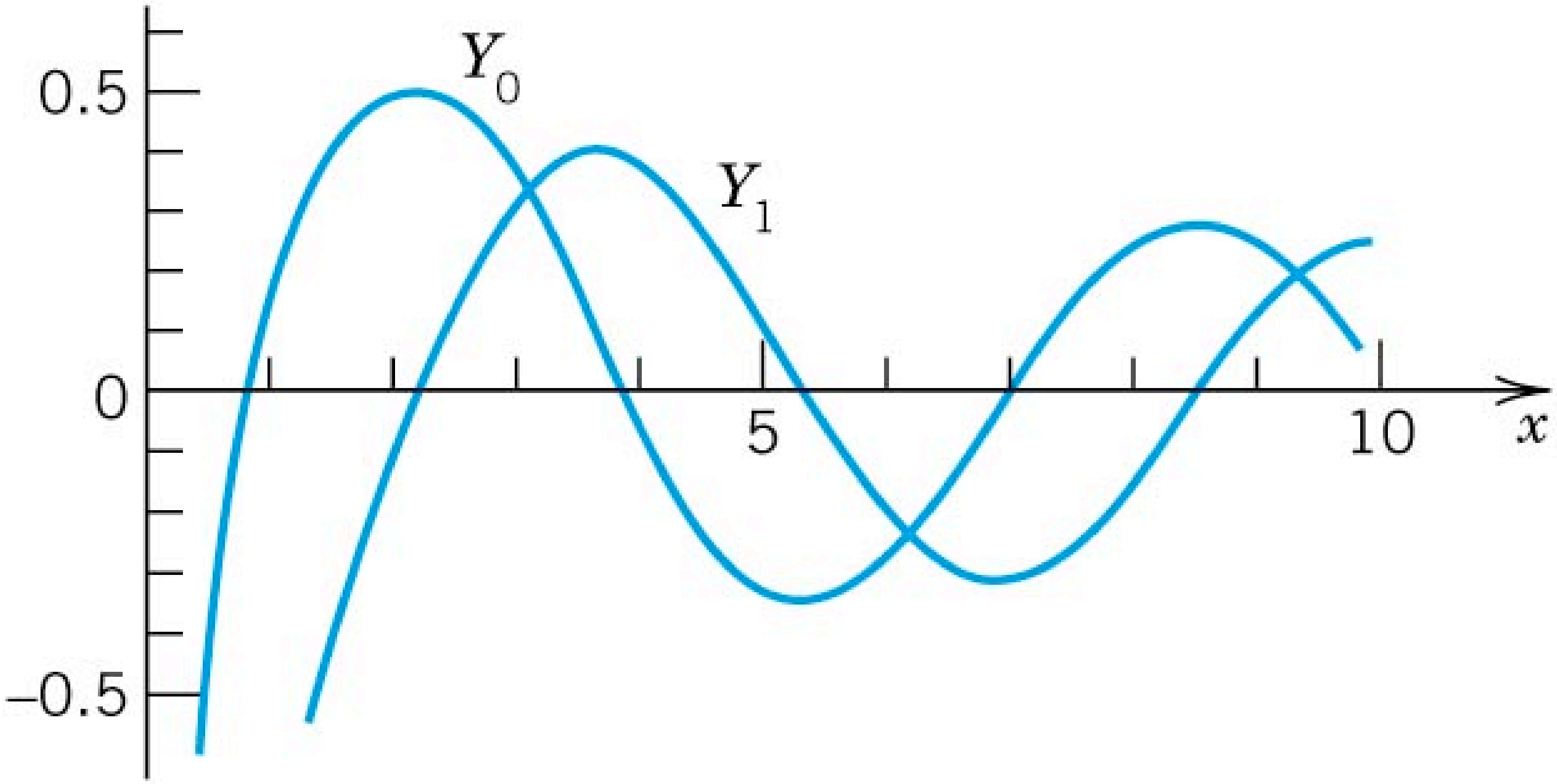


Fig. 105. Bessel functions of the second kind.
(For a small table, see Appendix 5.)

Table A1 *(continued)*

x	$Y_0(x)$	$Y_1(x)$	x	$Y_0(x)$	$Y_1(x)$	x	$Y_0(x)$	$Y_1(x)$
0.0	$(-\infty)$	$(-\infty)$	2.5	0.498	0.146	5.0	−0.309	0.148
0.5	−0.445	−1.471	3.0	0.377	0.325	5.5	−0.339	−0.024
1.0	0.088	−0.781	3.5	0.189	0.410	6.0	−0.288	−0.175
1.5	0.382	−0.412	4.0	−0.017	0.398	6.5	−0.173	−0.274
2.0	0.510	−0.107	4.5	−0.195	0.301	7.0	−0.026	−0.303

Table A2
Gamma Function [see (24) in Appendix A3.1]

α	$\Gamma(\alpha)$	α	$\Gamma(\alpha)$	α	$\Gamma(\alpha)$	α	$\Gamma(\alpha)$	α	$\Gamma(\alpha)$
1.00	1.000 000	1.20	0.918 169	1.40	0.887 264	1.60	0.893 515	1.80	0.931 384
1.02	0.988 844	1.22	0.913 106	1.42	0.886 356	1.62	0.895 924	1.82	0.936 845
1.04	0.978 438	1.24	0.908 521	1.44	0.885 805	1.64	0.898 642	1.84	0.942 612
1.06	0.968 744	1.26	0.904 397	1.46	0.885 604	1.66	0.901 668	1.86	0.948 687
1.08	0.959 725	1.28	0.900 718	1.48	0.885 747	1.68	0.905 001	1.88	0.955 071
1.10	0.951 351	1.30	0.897 471	1.50	0.886 227	1.70	0.908 639	1.90	0.961 766
1.12	0.943 590	1.32	0.894 640	1.52	0.887 039	1.72	0.912 581	1.92	0.968 774
1.14	0.936 416	1.34	0.892 216	1.54	0.888 178	1.74	0.916 826	1.94	0.976 099
1.16	0.929 803	1.36	0.890 185	1.56	0.889 639	1.76	0.921 375	1.96	0.983 743
1.18	0.923 728	1.38	0.888 537	1.58	0.891 420	1.78	0.926 227	1.98	0.991 708
1.20	0.918 169	1.40	0.887 264	1.60	0.893 515	1.80	0.931 384	2.00	1.000 000

Table A3
Factorial Function

n	$n!$	log ($n!$)	n	$n!$	log ($n!$)	n	$n!$	log ($n!$)
1	1	0.000 000	6	720	2.857 332	11	39 916 800	7.601 156
2	2	0.301 030	7	5 040	3.702 431	12	479 001 600	8.680 337
3	6	0.778 151	8	40 320	4.605 521	13	6 227 020 800	9.794 280
4	24	1.380 211	9	362 880	5.559 763	14	87 178 291 200	10.940 408
5	120	2.079 181	10	3 628 800	6.559 763	15	1 307 674 368 000	12.116 500

Table A4
Error Function, Sine and Cosine Integrals [see (35), (40), (42) in Appendix A3.1]

x	erf x	Si(x)	ci(x)	x	erf x	Si(x)	ci(x)
0.0	0.0000	0.0000	∞	2.0	0.9953	1.6054	−0.4230
0.2	0.2227	0.1996	1.0422	2.2	0.9981	1.6876	−0.3751
0.4	0.4284	0.3965	0.3788	2.4	0.9993	1.7525	−0.3173
0.6	0.6039	0.5881	0.0223	2.6	0.9998	1.8004	−0.2533
0.8	0.7421	0.7721	−0.1983	2.8	0.9999	1.8321	−0.1865
1.0	0.8427	0.9461	−0.3374	3.0	1.0000	1.8487	−0.1196
1.2	0.9103	1.1080	−0.4205	3.2	1.0000	1.8514	−0.0553
1.4	0.9523	1.2562	−0.4620	3.4	1.0000	1.8419	0.0045
1.6	0.9763	1.3892	−0.4717	3.6	1.0000	1.8219	0.0580
1.8	0.9891	1.5058	−0.4568	3.8	1.0000	1.7934	0.1038
2.0	0.9953	1.6054	−0.4230	4.0	1.0000	1.7582	0.1410

5.2 Bessel's Equations of Order ν with Parameter λ

$$x^2 y'' + x y' + (\lambda^2 x^2 - \nu^2) y = 0$$

Let $t = \lambda x$

$$dy/dx = \lambda\ dy/dt, \quad d^2y/dx^2 = \lambda^2 d^2y/dt^2$$

$\therefore$
$$t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - \nu^2) y = 0$$

$\Rightarrow$
$$y = c_1 J_\nu(t) + c_2 Y_\nu(t)$$

or
$$y = c_1 J_\nu(\lambda x) + c_2 Y_\nu(\lambda x)$$

If ν is not a positive integer or 0, then the solution can be also written as

$$y = c_1 J_\nu(\lambda x) + c_2 J_{-\nu}(\lambda x)$$

5.3 Modified Bessel's Functions

$$x^2 y'' + x y' - (x^2 + \nu^2) y = 0 \qquad (1)$$

Note that the solution of $x^2 y'' + x y' + (\lambda^2 x^2 - \nu^2) y = 0$ is given by

$$y = c_1 J_\nu(\lambda x) + c_2 Y_\nu(\lambda x)$$

In this case, $\lambda = i$, $\lambda^2 = -1$, thus the solution of (1) is

$$y = c_1 J_\nu(ix) + c_2 Y_\nu(ix)$$

But

$$J_\nu(ix) = \sum_{m=0}^{\infty} \frac{(-1)^m (ix)^{\nu+2m}}{2^{\nu+2m} m!\, \Gamma(\nu + m + 1)}$$

$$= i^\nu \sum_{m=0}^{\infty} \frac{x^{\nu+2m}}{2^{\nu+2m} m!\, \Gamma(\nu + m + 1)}$$

∴ we let

$I_\nu(x) \equiv i^{-\nu} J_\nu(ix)$ = Modified Bessel's Function of the First Kind of Order ν

If ν is not an integer, the general solution of

$$x^2 y'' + x y' - (x^2 + \nu^2) y = 0$$

is given by

$$y = c_1 I_\nu(x) + c_2 I_{-\nu}(x)$$

If $\nu = n$, an integer, then

$$I_n(x) = I_{-n}(x)$$

We now define

$$K_\nu(x) \equiv \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin\nu\pi}$$

= Modified Bessel's Function of the second kind with order ν

For $\nu = n$, $K_n(x) = \lim_{\nu \to n} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin\nu\pi}$

In summary, the general solution of

$$x^2 y'' + x y' - (x^2 + n^2) y = 0$$

is $$y = c_1 I_n(x) + c_2 K_n(x)$$

and that the general solution of

$$x^2 y'' + x y' - (x^2 + \nu^2) y = 0 \quad , \nu \notin I$$

is given by

$$y = c_1 I_\nu(x) + c_2 I_{-\nu}(x)$$

or $$y = c_1 I_\nu(x) + c_2 K_\nu(x)$$

Note that $I_n(\infty) = \infty$ $\qquad K_n(\infty) = 0$

$I_0(0) = 1$ $\qquad K_n(0) = \infty$

$I_1(0) = 0$

5.4 Equations Solvable in Terms of Bessel Functions

(i) **If $(1-a)^2 \geq 4c$ and if neither d, p nor q is zero, then** (except the Euler Equation),

$$x^2 y'' + x(a + 2bx^p)y' + [c + dx^{2q} + b(a + r - 1)x^p + b^2 x^{2p}]y = 0$$

has the complete solution

$$y = x^{\alpha} e^{-\beta x^p} [c_1 J_\nu(\lambda x^q) + c_2 Y_\nu(\lambda x^q)]$$

where

$$\alpha = \frac{1-a}{2}, \quad \beta = \frac{b}{p}$$

$$\lambda = \frac{\sqrt{|d|}}{q}, \quad \nu = \frac{\sqrt{(1-a)^2 - 4c}}{2q}$$

Note p=r and q=s in the other course notes.

If $d < 0$, J_ν and Y_ν are to be replaced by I_ν and K_ν, respectively.

(ii) If $(1 - r)^2 \geq 4b$, $a \neq 0$ and if either $r - 2 < s$ or $b = 0$, then (except for Euler Equation)

$$(x^r y')' + (a x^s + b x^{r-2}) y = 0$$

has a complete solution

$$y = x^{\alpha} [c_1 J_{\nu}(\lambda x^{r}) + c_2 Y_{\nu}(\lambda x^{r})]$$

where

$$\alpha = \frac{1-r}{2} \quad , \quad \gamma = \frac{2-r+s}{2}$$

$$\lambda = \frac{2\sqrt{|a|}}{2-r+s} \quad , \quad \nu = \frac{\sqrt{(1-r)^2 - 4b}}{2-r+s}$$

If $a < 0$, J_ν and Y_ν are to be replaced by I_ν and K_ν, respectively.

(iii) $$x^2 y'' + a x y' + (b x^c + d) y = 0$$

$$\Rightarrow \quad y = c_1 x^{\alpha} J_{\nu}(\lambda x^{\beta}) + c_2 x^{\alpha} Y_{\nu}(\lambda x^{\beta})$$

where

$$\alpha = \frac{1 - a}{2} \qquad \beta = \frac{c}{2}$$

$$\lambda^2 = \frac{4b}{c^2} \qquad \nu^2 = \frac{4(\alpha^2 - d)}{c^2}$$

[Example] $x^2 y'' + a x y' + (b x^c + d) y = 0$

[Solution] Let $y = u x^{\alpha}$

$$y' = \alpha u x^{\alpha-1} + u' x^{\alpha}$$

$$y'' = \alpha (\alpha - 1) u x^{\alpha-2} + \alpha u' x^{\alpha-1} + \alpha u' x^{\alpha-1} + u'' x^{\alpha}$$

$$= u'' x^{\alpha} + 2 \alpha x^{\alpha-1} u' + \alpha (\alpha - 1) u x^{\alpha-2}$$

Thus, the differential equation becomes

$$x^2 (u'' x^{\alpha} + 2 \alpha x^{\alpha-1} u' + \alpha (\alpha - 1) u x^{\alpha-2})$$

$$+ a x (\alpha u x^{\alpha-1} + u' x^{\alpha}) + (b x^c + d) u x^{\alpha} = 0$$

or $$x^2 u'' + (2 \alpha + a) x u' + \{ b x^c + \alpha (\alpha - 1) + a \alpha + d \} u = 0$$

Compare the above equation with the standard form of the Bessel's differential equation

$$x^2 y'' + x y' + (\lambda^2 x^2 - \nu^2) y = 0$$

we set $2 \alpha + a = 1$

$\Rightarrow$ $$x^2 u'' + x u' + \{ b x^c + d - \alpha^2 \} u = 0$$

Next, we set

$$z^2 = x^c \quad \text{or} \quad z = x^{c/2}$$

then $$u' = \frac{du}{dx} = \frac{du}{dz} \frac{dz}{dx} = \frac{du}{dz} \frac{c}{2} x^{\frac{c}{2} - 1}$$

$$u'' = \frac{d}{dx}\frac{du}{dx} = \frac{d}{dx}\left[\frac{du}{dz}\frac{c}{2} x^{\frac{c}{2}-1} \right]$$

$$= \frac{d^2u}{dxdz}\frac{c}{2}\, x^{\frac{c}{2}-1} + \frac{du}{dz}\frac{c}{2}\left[\frac{c}{2}-1\right] x^{\frac{c}{2}-2}$$

$$= \frac{d^2u}{dz^2}\left[\frac{c}{2}\right]^2 x^{2\left[\frac{c}{2}-1\right]} + \frac{du}{dz}\frac{c}{2}\left[\frac{c}{2}-1\right] x^{\frac{c}{2}-2}$$

After substitution u' and u'' into the differential equation, we have

$$z^2\frac{d^2u}{dz^2} + z\frac{du}{dz} + \left\{ \frac{4\,b}{c^2} z^2 - \frac{4\,(-d+\alpha^2)}{c^2} \right\} u = 0$$

Again, compare with the standard Bessel's differential equation, we can set

$$\lambda^2 = \frac{4\,b}{c^2} \quad ; \quad \nu^2 = \frac{4\,(\alpha^2 - d)}{c^2}$$

and the solution is thus

$$u = c_1 J_\nu(\lambda z) + c_2 Y_\nu(\lambda z)$$

but $\quad y = u\,x^\alpha \,,\; z = x^{c/2} = x^\beta$

$\therefore \quad y = c_1 x^\alpha J_\nu(\lambda x^\beta) + c_2 x^\alpha Y_\nu(\lambda x^\beta)$

where $\quad \alpha = \dfrac{1-a}{2} \qquad\qquad (a = 1 - 2\alpha)$

$$\beta = \frac{c}{2}$$

$$\lambda^2 = \frac{4\,b}{c^2} \qquad\qquad (\,\lambda^2 \beta^2 = b\,)$$

$$\nu^2 = \frac{4\,(\,\alpha^2 - d\,)}{c^2} \qquad\qquad (\,\alpha^2 - \nu^2 \beta^2 = d\,)$$

[Exercise] Try to use the substitutions

$$y = u\,x^{1/2} \text{ and } z = x^{1/2}$$

to find the solutions (in terms of Bessel's functions) to the following differential equation

$$x\,y'' + y = 0$$

[Answer]

$$y = c_1 \sqrt{x}\; J_1(\,2\sqrt{x}\,) + c_2 \sqrt{x}\; Y_1(\,2\sqrt{x}\,)$$

5.5 Some Important Properties of Bessel Functions

(A) Identities

(1) $$\frac{d[\, x^{\nu} J_{\nu}(x)\,]}{dx} = x^{\nu} J_{\nu-1}(x)$$

(2) $$\frac{d[\, x^{-\nu} J_{\nu}(x)\,]}{dx} = - x^{-\nu} J_{\nu+1}(x)$$

(1)$_Y$ $$\frac{d[\, x^{\nu} Y_{\nu}(x)\,]}{dx} = x^{\nu} Y_{\nu-1}(x)$$

(2)$_Y$ $$\frac{d[\, x^{-\nu} Y_{\nu}(x)\,]}{dx} = - x^{-\nu} Y_{\nu+1}(x)$$

(1)$_I$ $$\frac{d[\, x^{\nu} I_{\nu}(x)\,]}{dx} = x^{\nu} I_{\nu-1}(x)$$

(2)$_I$ $$\frac{d[\, x^{-\nu} I_{\nu}(x)\,]}{dx} = x^{-\nu} I_{\nu+1}(x)$$

(1)$_K$ $$\frac{d[\, x^{\nu} K_{\nu}(x)\,]}{dx} = - x^{\nu} K_{\nu-1}(x)$$

(2)$_K$ $$\frac{d[\, x^{-\nu} K_{\nu}(x)\,]}{dx} = - x^{-\nu} K_{\nu+1}(x)$$

(3) $$\boxed{J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_{\nu}(x)}$$

$$(4) \qquad J_{\nu-1}(x) - J_{\nu+1}(x) \;=\; 2\,J_\nu'\,(x)$$

$$\boxed{\begin{aligned} &(5) \qquad \int x^{\nu} J_{\nu-1}(x)\,dx \;=\; x^{\nu} J_\nu(x) + C \\ &(6) \qquad \int x^{-\nu} J_{\nu+1}(x)\,dx \;=\; -\,x^{-\nu} J_\nu(x) + C \end{aligned}}$$

$$(7) \qquad \int J_{\nu+1}(x)\,dx \;=\; \int J_{\nu-1}(x)\,dx - 2\,J_\nu(x)$$

[Example] Express $J_4(\lambda x)$ in terms of $J_0(\lambda x)$ and $J_1(\lambda x)$

[Solution] It is known that

$$J_{\nu-1}(\lambda x) + J_{\nu+1}(\lambda x) = \frac{2\nu}{\lambda x} J_\nu(\lambda x)$$

$$\therefore \quad J_{\nu+1}(\lambda x) = \frac{2\nu}{\lambda x} J_\nu(\lambda x) - J_{\nu-1}(\lambda x)$$

$$\Rightarrow \quad J_4(\lambda x) = \frac{6}{\lambda x} J_3(\lambda x) - J_2(\lambda x)$$

$$J_3(\lambda x) = \frac{4}{\lambda x} J_2(\lambda x) - J_1(\lambda x)$$

$$J_2(\lambda x) = \frac{2}{\lambda x} J_1(\lambda x) - J_0(\lambda x)$$

$$\Rightarrow \quad J_4(\lambda x) = \left\{ \frac{48}{\lambda^3 x^3} - \frac{8}{\lambda x} \right\} J_1(\lambda x) - \left\{ \frac{24}{\lambda^2 x^2} - 1 \right\} J_0(\lambda x)$$

[Exercise] Show that

$$\int x\, J_0(x)\, dx = x\, J_1(x) + C$$

$$\because \frac{d}{dx}\left[x J_1(x) \right] = x J_0(x)$$

$$\int J_1(x)\, dx = -J_0(x) + C$$

$$\because \frac{dJ_0(x)}{dx} = -J_1(x)$$

$$\boxed{\frac{d}{dx}\left[x^{-\nu} J_\nu(x) \right] = -x^{-\nu} J_{\nu+1}(x)}$$

$$\frac{d}{dx}\left[x^{-\nu}J_\nu(x)\right] = -x^{-\nu}J_{\nu+1}(x)$$

[Example] Evaluate $\int J_3(x)\, dx$

[Solution] $\int J_3(x)\, dx = \int x^2 \left[x^{-2} J_3(x) \right] dx$

$$\underset{u}{\Downarrow} \qquad \underset{dv}{\Downarrow}$$

$$u = x^2 \quad ; \quad dv = x^{-2} J_3(x)\, dx = d(-x^{-2} J_2(x))$$

$$\nu = 2$$

Integration by parts:

$$\int J_3(x)\, dx = -x^2 (x^{-2} J_2(x)) + \int x^{-2} J_2(x)\, dx^2$$

$$= -J_2(x) + \int 2x^{-1} J_2(x)\, dx \qquad \nu = 1$$

$$= -J_2(x) - 2x^{-1} J_1(x) + C$$

[Example] $\displaystyle\int \frac{J_2(3x)}{x^2}\,dx = \frac{1}{9}\int [\,(3x)^2 J_2(3x)\,]\,\frac{dx}{x^4}$

$$\Downarrow \qquad \Downarrow$$
$$u \qquad dv$$

$\dfrac{d}{dx}\left[x^{\nu} J_{\nu}(x)\right] = x^{\nu} J_{\nu-1}(x)$ $\quad\therefore\quad du = (3x)^2 J_1(3x)\,3\,dx \qquad v = -\dfrac{1}{3x^3}$

$$\therefore \quad \int \frac{J_2(3x)}{x^2}\,dx = \frac{1}{9}\left\{-\frac{3J_2(3x)}{x} + \int \frac{1}{3x^3}(3x)^2 J_1(3x)\,3\,dx\right\}$$

$$= \frac{1}{9}\left\{-\frac{3J_2(3x)}{x} + 3\int 3x J_1(3x)\,\frac{dx}{x^2}\right\}$$

$$\Downarrow \qquad \Downarrow$$
$$u \qquad dv$$

$$u = 3x J_1(3x) \quad ; \quad dv = dx/x^2$$

$\dfrac{d}{dx}\left[x^{\nu} J_{\nu}(x)\right] = x^{\nu} J_{\nu-1}(x)$ $\quad\therefore\quad du = 3x J_0(3x)\,3\,dx\,; \qquad v = -1/x$

$$\therefore \quad \int \frac{J_2(3x)}{x^2}\,dx = \frac{1}{9}\left\{-\frac{3J_2(3x)}{x} + 3\left[-3J_1(3x) + 9\int J_0(3x)\,dx\right]\right\}$$

$$= -\frac{J_2(3x)}{3x} - J_1(3x) + 3\int J_0(3x)\,dx$$

In general, an integral of the form

$$\int x^m J_n(x)\,dx \qquad m+n \geq 0$$

1. completely integrated if $m + n$ = odd,
2. have $\int J_0(x)\,dx$ for $m + n$ = even.

(B) Behavior Near the Origin

$n = 0$ $\quad J_0(0) = I_0(0) = 1$

$$Y_0(0) = -\infty$$

$$K_0(0) = \infty$$

$n = 1, 2, \ldots$ $\quad J_n(0) = I_n(0) = 0$

$$Y_n(0) = -\infty$$

$$K_n(0) = \infty$$

(C) Asymptotic Behavior for Large x

$$J_n(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4} - \frac{n\pi}{2} \right)$$

$$Y_n(x) \approx \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi}{4} - \frac{n\pi}{2} \right)$$

(D) Bessel Function of Half Integer Order

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sinh x$$

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \cosh x$$

(E) Bessel Function of Negative Order, $n \in N$

$$J_{-n}(x) = (-1)^n J_n(x)$$
$$Y_{-n}(x) = (-1)^n Y_n(x)$$
$$I_{-n}(x) = I_n(x)$$
$$K_{-n}(x) = K_n(x)$$

Summary

1 $$f(x) = \sum_{m=0}^{\infty} \frac{f^{(m)}(x_0)}{m!} (x - x_0)^m$$ Taylor Series

When $x_0 = 0$ $\Rightarrow$ *Maclaurin Series*

2 *Ratio Test*

$$\rho = \lim_{m\to\infty} \left| \frac{a_{m+1} (x - x_0)^{m+1}}{a_m (x - x_0)^m} \right|$$

3 Analytic Function, Regular Point, Singular Point, etc.

$$y'' + p(x)\, y' + q(x)\, y = 0$$

If $p(x)$, $q(x)$ are analytic at $x = 0$ $\Rightarrow$ $x = 0$ is a regular point

$$\Rightarrow y = \sum_{m=0}^{\infty} a_m x^m$$

If $p(x)$, $q(x)$ are not analytic at $x = 0$ $\Rightarrow$ singular point

For $x = 0$ is a singular point, rewrite the differential equation in the following form:

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0$$

If $b(x)$, $c(x)$ analytic at $x = 0$ $\Rightarrow$ regular singular point

$$\Rightarrow \ y = x^r \sum_{m=0}^{\infty} a_m x^m$$

If b(x), c(x) not analytic $\Rightarrow$ irregular singular point

4 Frobenius Method - Extended Power Series Method

Any differential equation of the form

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0$$

where b(x) and c(x) are analytic at x = 0, has at least one solution of the form

$$y = x^r \sum_{m=0}^{\infty} a_m x^m = x^r (a_0 + a_1 x + a_2 x^2 + \ \dots), \ a_0 \neq 0$$

where r may be any number (real or complex).

<u>Form of the Second Solution</u>

Case 1: r_1 and r_2 differ but not by an integer

$$y_1 = x^{r_1} (a_0 + a_1 x + a_2 x^2 + \dots)$$

$$y_2 = x^{r_2} (A_0 + A_1 x + A_2 x^2 + \dots)$$

Case 2: $r_1 = r_2 = r, \quad r = \frac{1}{2} (1 - b_0)$

$$y_1 = x^r (a_0 + a_1 x + a_2 x^2 + \dots)$$

$$y_2 = y_1 \ln x + x^r (A_1 x + A_2 x^2 + \ldots)$$

Case 3: r_1 and r_2 differ by a nonzero integer, where $r_1 > r_2$

$$y_1 = x^{r_1} (a_0 + a_1 x + a_2 x^2 + \ldots)$$

$$y_2 = k\, y_1 \ln x + x^{r_2} (A_0 + A_1 x + A_2 x^2 + \ldots)$$

where $r_1 - r_2 > 0$ and k <u>may</u> be zero.

Note that in Case 2 and Case 3, the second linearly independent solution y_2 can also be obtained by reduction of order method (i.e., by assuming $y_2 = u\, y_1$).

1. Legendre's Differential Equation

$$(1 - x^2) y'' - 2 x y' + n (n + 1) y = 0 , \quad n = 0, 1, 2, \ldots$$

$$y = c_1 P_n(x) + c_2 Q_n(x)$$

where $P_n(x)$ = Legendre polynomials
$Q_n(x)$ = Legendre functions of the second kind

2. $x^2 y'' + x y' + (x^2 - \nu^2) y = 0$

(1) $\nu \in N$ ($\nu = n$)

$$y = c_1 J_n(x) + c_2 Y_n(x)$$
$$y = c_1 J_n(x) + c_2 J_{-n}(x) \quad \Leftarrow \quad \text{No!}$$

(2) $\nu \notin N$

$$y = c_1 J_\nu(x) + c_2 Y_\nu(x)$$
or $$y = c_1 J_\nu(x) + c_2 J_{-\nu}(x)$$

Need to specify J and Y . . .

3. $x^2 y'' + x y' + (\lambda^2 x^2 - \nu^2) y = 0$

(1) $\nu \in N$ ($\nu = n$)

$$y = c_1 J_n(\lambda x) + c_2 Y_n(\lambda x)$$
$$y = c_1 J_n(\lambda x) + c_2 J_{-n}(\lambda x) \quad \Leftarrow \quad \text{No!}$$

(2) $\nu \notin N$

$$y = c_1 J_\nu(\lambda x) + c_2 Y_\nu(\lambda x)$$
or $$y = c_1 J_\nu(\lambda x) + c_2 J_{-\nu}(\lambda x)$$

Need to specify J and Y . . .

4. $x^2 y'' + x y' - (x^2 + \nu^2) y = 0$

(1) $\nu \in N$ ($\nu = n$)

$$y = c_1 I_n(x) + c_2 K_n(x)$$
$$y = c_1 I_n(x) + c_2 I_{-n}(x) \quad \Leftarrow \quad \text{No!}$$

(2) $\nu \notin N$

$$y = c_1 I_\nu(x) + c_2 K_\nu(x)$$
or $$y = c_1 I_\nu(x) + c_2 I_{-\nu}(x)$$

Need to specify I and K . . .

5. J_0, J_1, Y_0, Y_1, I_0, I_1, K_0, K_1 之圖形